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Three neutrino oscillations in matter

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ABSTRACT

Following similar approaches in the past, the Schrödinger equation for three neutrino propagation in matter of constant density is solved analytically by two successive diagonalizations of 2×2 matrices. The final result for the oscillation probabilities is obtained directly in the conventional parametric form as in the vacuum but with explicit simple modification of two mixing angles (θ_{12} and θ_{13}) and mass eigenvalues. In this form, the analytical results provide excellent approximation to numerical calculations and allow for simple qualitative understanding of the matter effects.

The MSW effect [1] for the neutrino propagation in matter attracts a lot of experimental and theoretical attention. Most recently, the discussion is focused on the DUNE experiment [2].

On the theoretical side, a large number of numerical simulations of the MSW effect in matter with a constant or varying density has been performed. Although, in principle, sufficient for comparing the theory predictions with experimental data, they do not provide a transparent physical interpretation of the experimental results. Therefore, several authors have also published analytical or semi-analytical solutions to the Schrödinger equation for three neutrino propagation in matter of constant density, in various perturbative expansions [3–5]. The complexity of the calculation, the transparency of the final result and the range of its applicability depend on the chosen expansion parameter.

In this short note we solve the Schrödinger equation in matter with constant density, using the approximate see-saw structure of the full Hamiltonian in the electroweak basis. This way one can diagonalize the 3×3 matrix by two successive diagonalizations of 2×2 matrices (similar approaches have been used in the past, in particular in Refs. [4] and [5]). We specifically have in mind the parameters of the DUNE experiment but our method is applicable for their much wider range. The final result for the oscillation probabilities is obtained directly in the conventional parametric form as in the vacuum but with modified two mixing angles and mass eigenvalues,[1] similarly to the well known results for the two-neutrino propagation in matter. The three neutrino oscillation probabilities in matter have been presented in the same form as here in the recent Ref. [6], where the earlier results obtained in Ref. [5] are rewritten in this form. The form of our final results can also be obtained after some simplifications from Ref. [4]. Our approach can be easily generalized to non-constant matter density by dividing the path of the neutrino trajectory in the matter to layers and assuming constant density in each layer.

The starting point is the Schrödinger equation

$$i\frac{d}{dx}\nu = \mathcal{H}\nu \tag{1}$$

where $\mathcal{H}$ is the Hamiltonian in matter. In the electroweak basis it reads

$$\mathcal{H} = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m^2_\odot}{2E} & 0 \\ 0 & 0 & \frac{\Delta m^2_a}{2E} \end{pmatrix} U^\dagger + \begin{pmatrix} V(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

The matrix U is the neutrino mixing matrix in the vacuum. The mass squared differences are defined as $\Delta m^2_\odot \equiv m_2^2 - m_1^2$ ($\approx 7.5 \times 10^{-5}$ eV^2) and $\Delta m^2_a \equiv m_3^2 - m_1^2$ ($\approx \pm 2.5 \times 10^{-3}$ eV^2, positive sign is for normal mass ordering and negative sign for inverted one). Here $V(x)$ is the neutrino weak interaction potential energy $V = \sqrt{2} G_F N_e$ (N_e is electron number density) and we take it in this section to be x-independent. The neutrino oscillation probabilities are determined by the S-matrix elements

$$S_{\alpha\beta} = T\, e^{-i\int_{x_0}^{x_f} \mathcal{H}(x)dx} \tag{3}$$

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[1] The results of this paper have been presented as private communication by one of us (A.I.) to the members of the T2HKK collaboration in December 2017.

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For a constant V and in order to obtain our results in the same form as for the oscillation probabilities in the vacuum, it is convenient to rewrite the S-matrix elements as follows:

$$S_{\alpha\beta} = e^{-iU_m \mathcal{H}_m U_m^\dagger (x_f - x_0)} = U_m e^{-i\mathcal{H}_m L} U_m^\dagger \tag{4}$$

The matrix $\mathcal{H}_m$ is the Hamiltonian in matter in the mass eigenstate basis:

$$\begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix} \tag{5}$$

and the U_m is the neutrino mixing matrix in matter. Defining $\phi_{21} = (\mathcal{H}_2 - \mathcal{H}_1)L$ and $\phi_{31} = (\mathcal{H}_3 - \mathcal{H}_1)L$, we can write

$$S_{\alpha\beta} = \left[U_m \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\phi_{21}} & 0 \\ 0 & 0 & e^{-i\phi_{31}} \end{pmatrix} U_m^\dagger \right] \tag{6}$$

Here we neglect irrelevant overall phase, $e^{-i\mathcal{H}_1 L}$. The neutrino transition probabilities do not depend on the overall phase of the S matrix.

The remaining task is to find the eigenvalues of $\mathcal{H}$ and the mixing matrix U_m:

$$\mathcal{H} = U_m \mathcal{H}_m U_m^\dagger \tag{7}$$

It is convenient to do it in two steps, first calculating the hamiltonian in a certain auxiliary basis. This way, to an excellent approximation, we can diagonalize the 3×3 matrix by two successive diagonalizations of the 2×2 matrices.

The auxiliary basis [7,8] is defined by the following equation

$$\mathcal{H}' = U^{aux\dagger} \mathcal{H} U^{aux} \quad \text{and} \quad S = U^{aux} e^{(-i\mathcal{H}' L)} U^{aux\dagger} \tag{8}$$

where

$$U^{aux} = \mathcal{O}_{23} U^\delta \mathcal{O}_{13} \tag{9}$$

and the rotations $\mathcal{O}_{ij}$ are defined by the decomposition of the mixing matrix U in the vacuum (see eq. (2)) as follows:

$$U = \mathcal{O}_{23} U^\delta \mathcal{O}_{13} U^{\delta *} \mathcal{O}_{12} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \tag{10}$$

where

$$U^\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \tag{11}$$

($c_{12} \equiv \cos\theta_{12}$, $s_{12} \equiv \sin\theta_{12}$ etc.).

The matrices $\mathcal{O}_{ij}$ are orthogonal matrices. It is more convenient to rewrite the matrix U in another form

$$U \to \tilde{U} = U \cdot U^\delta = \mathcal{O}_{23} U^\delta \mathcal{O}_{13} \mathcal{O}_{12} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23}e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23}e^{i\delta} \end{pmatrix} \tag{12}$$

Using eqs. (2), (8) we obtain

$$\mathcal{H}' = \mathcal{O}_{13}^T U^{\delta *} \mathcal{O}_{23}^T \, \mathcal{H} \, \mathcal{O}_{23} U^\delta \mathcal{O}_{13} = \begin{pmatrix} V c_{13}^2 & s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} & s_{13}c_{13}V \\ s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} & (c_{12}^2 - s_{12}^2)\frac{\Delta m_\odot^2}{2E} & 0 \\ s_{13}c_{13}V & 0 & \frac{\Delta m_{ee}^2}{2E} + V s_{13}^2 \end{pmatrix}, \tag{13}$$

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_a^2 + s_{12}^2 (\Delta m_a^2 - \Delta m_\odot^2) \tag{14}$$

The term $s_{12}^2 \frac{\Delta m_\odot^2}{2E}$ has been subtracted from the diagonal elements; it gives an overall phase to the S-matrix and according to the comments after eq. (6) is irrelevant.

The definition of Δm_{ee}^2 coincides with one of the definitions of the effective mass squared differences measured at reactor experiments [9,10].

This matrix has a see-saw structure, with the (13), (31) elements much smaller than the (33) element and can be put in an almost diagonal form by two rotations

$$\mathcal{O}_{12}^{m\,T} \mathcal{O}_{13}^{\prime\,T} \mathcal{H}' \, \mathcal{O}_{13}' \mathcal{O}_{12}^m = \begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix} \tag{15}$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_{31}^2}{2E} \end{pmatrix} + \mathcal{H}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

After the first rotation we have

$$\mathcal{O}_{13}^{\prime T} \mathcal{H}' \mathcal{O}_{13}' = \begin{pmatrix} \sin^2\theta_{13}' \frac{\Delta m_{ee}^2}{2E} + \cos^2(\theta_{13} + \theta_{13}')V & \cos\theta_{13}' \, s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} & 0 \\ \cos\theta_{13}' \, s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} & (c_{12}^2 - s_{12}^2)\frac{\Delta m_\odot^2}{2E} & \sin\theta_{13}' \, s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} \\ 0 & \sin\theta_{13}' \, s_{12}c_{12}\frac{\Delta m_\odot^2}{2E} & \cos^2\theta_{13}' \frac{\Delta m_{ee}^2}{2E} + \sin^2(\theta_{13} + \theta_{13}')V \end{pmatrix} \tag{17}$$

where

$$\sin 2\theta_{13}' = \frac{\epsilon_a \sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \tag{18}$$

and

$$\epsilon_a = \frac{2EV}{\Delta m_{ee}^2} \tag{19}$$

We can safely neglect the (23), (32) elements which are generated after the first rotation (see Appendix A) and diagonalize the remaining 2×2 sub-matrix with the second rotation

$$\sin 2\theta_{12}^m = \frac{\cos\theta_{13}' \sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2\theta_{13}' \sin^2 2\theta_{12}}},$$

$$\text{where} \quad \epsilon_\odot = \frac{2EV}{\Delta m_\odot^2}\left(\cos^2(\theta_{13} + \theta_{13}') + \frac{\sin^2\theta_{13}'}{\epsilon_a}\right). \tag{20}$$

The eigenvalues of $\mathcal{H}$ are

$$\mathcal{H}_2 - \mathcal{H}_1 \equiv \frac{\Delta m_{21}^2}{2E} = \frac{\Delta m_\odot^2}{2E}\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2\theta_{13}' \sin^2 2\theta_{12}}, \tag{21}$$