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Rastall gravity is equivalent to Einstein gravity

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A B S T R A C T

Rastall gravity, originally developed in 1972, is currently undergoing a significant surge in popularity. Rastall gravity purports to be a modified theory of gravity, with a non-conserved stress–energy tensor, and an unusual non-minimal coupling between matter and geometry, the Rastall stress–energy satisfying $[T_R]^{ab}{}_{;b} = \frac{\lambda}{4} g^{ab} R_{;b}$. Unfortunately, a deeper look shows that Rastall gravity is completely equivalent to Einstein gravity — usual general relativity. The gravity sector is completely standard, based as usual on the Einstein tensor, while in the matter sector Rastall's stress–energy tensor corresponds to an artificially isolated *part* of the physical conserved stress–energy.

1. Introduction

Rastall gravity [1], despite its somewhat mixed 45-year history, is currently undergoing a significant surge in popularity. Some 19 closely related articles have appeared so far in 2017 [2–20]. (See also [21–23].)

Unfortunately, as I shall argue below, Rastall gravity is completely equivalent to standard Einstein gravity — general relativity — all that is going on is that one is artificially splitting the physical conserved stress–energy tensor into two non-conserved pieces.

Historically, in 1972 Rastall tentatively suggested [1] that it might prove profitable to consider a covariantly non-conserved stress–energy tensor, one with $\nabla_b [T_R]^{ab} \neq 0$. In particular, he then suggested the phenomenological model $\nabla_b [T_R]^{ab} = F^a$, where F^a is some vector field vanishing in flat spacetime.

A (somewhat weak) plausibility argument then led him to consider $\nabla_b [T_R]^{ab} \propto g^{ab} \nabla_a R$. Ultimately Rastall posited the existence of a constant λ such that for Rastall's non-conserved stress energy tensor

$$\nabla_b [T_R]^{ab} = \frac{\lambda}{4} g^{ab} \nabla_b R. \tag{1}$$

(For future convenience, I have chosen a slightly different normalization than Rastall.) The full Rastall equations of motion (EOM) are then [1]:

$$G_{ab} + \frac{1}{4} \lambda R g_{ab} = \kappa [T_R]_{ab}, \tag{2}$$

whence

$$(\lambda - 1) R = \kappa T_R. \tag{3}$$

So already at this stage it is clear that the case $\lambda = 1$ is special.

There are numerous and extensive claims in the literature that Rastall's approach amounts to introducing a deep non-minimal coupling between gravity and matter. Unfortunately, as we shall see below, in terms of the underlying physics, this approach proves simply to be a content-free rearrangement of the matter sector. As gravity, there is absolutely nothing new in this proposal.

Similar comments can be found in a little-known 1982 paper by Lindblom and Hiscock [24]. As per the discussion below, in this particular 35-year-old article the authors emphasize the construction of a conserved stress–energy tensor, algebraically built from the Rastall stress–energy [24].[1]

2. Rastall gravity in vacuum

First, we observe that in vacuum Rastall's equation reduces to

$$G_{ab} + \frac{1}{4} \lambda R g_{ab} = 0; \qquad (\lambda - 1) R = 0. \tag{4}$$

If $\lambda \neq 1$ this implies

[1] Where Lindblom and Hiscock differ from the current analysis is by introducing the explicit (and quite radical) assumption that laboratory equipment couples only to the non-conserved Rastall stress–energy, not to the conserved stress–energy tensor [24]. This allows them to place stringent phenomenological constraints on Rastall's λ parameter: $|\lambda| < 10^{-15}$. I will not be exploring this particular route in the current article.

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$$G_{ab} = 0; \tag{5}$$

whereas if $\lambda = 1$ one obtains

$$G_{ab} = \Lambda\, g_{ab}. \tag{6}$$

This is either the standard vacuum Einstein EOM, or at worst the Einstein EOM + (arbitrary cosmological constant). The vacuum solution is simply an Einstein spacetime. (For $\lambda \neq 1$ this vacuum degeneracy between the Rastall and Einstein theories was already noted by Rastall some 45 years ago [1].)

3. Adding matter: generic case ($\lambda \neq 1$)

Since $R = \frac{\kappa T_R}{\lambda - 1}$, we construct the geometrical Einstein tensor in terms of Rastall's stress–energy as

$$G_{ab} = \kappa \left([T_R]_{ab} + \frac{1}{4} \frac{\lambda}{1-\lambda} T_R\, g_{ab} \right). \tag{7}$$

Therefore, if we choose to define

$$T_{ab} = [T_R]_{ab} + \frac{1}{4} \frac{\lambda}{1-\lambda} T_R\, g_{ab}, \tag{8}$$

then this quantity is covariantly conserved. Thus it is this stress energy that should be considered physical, and in terms of this physical stress–energy tensor

$$G_{ab} = \kappa\, T_{ab} \tag{9}$$

is the usual Einstein equation.

We can of course invert this construction using

$$T = T_R + \frac{\lambda}{1-\lambda} T_R = \frac{1}{1-\lambda} T_R; \tag{10}$$

so that

$$T_R = (1-\lambda) T. \tag{11}$$

We see

$$[T_R]_{ab} = T_{ab} - \frac{1}{4} \lambda\, T\, g_{ab}. \tag{12}$$

That is, from the Rastall stress–energy $[T_R]_{ab}$, (and knowledge of the Rastall coupling λ), one can always reconstruct the physical stress–energy T_{ab}, *and vice versa*. So, (at least for $\lambda \neq 1$), all that is going on is that Rastall has simply mis-identified the physical stress–energy. In terms of the true physical conserved stress–energy T_{ab} one just has standard Einstein gravity.[2] Indeed, one can easily jump back and forth using equations (8) and (12). Sometimes this very simple observation is hidden very deeply in very technical, very specific, and very turgid calculations.[3]

4. Adding matter: special case ($\lambda = 1$)

This is the only case that is even mildly interesting. Ironically, it was already considered (and rejected) by Rastall 45 years ago [1]. For $\lambda = 1$ the Rastall EOM reduce to

$$G_{ab} + \frac{1}{4} R g_{ab} = \kappa [T_R]_{ab}; \qquad T_R = 0; \tag{13}$$

or alternatively

$$R_{ab} - \frac{1}{4} R g_{ab} = \kappa [T_R]_{ab}; \qquad T_R = 0. \tag{14}$$

So in this $\lambda = 1$ special case situation Rastall matter has to be traceless. In terms of the physical stress–energy this is simply

$$G_{ab} + \frac{1}{4} R\, g_{ab} = \kappa \left(T_{ab} - \frac{1}{4} T g_{ab} \right), \tag{15}$$

or alternatively

$$R_{ab} - \frac{1}{4} R\, g_{ab} = \kappa \left(T_{ab} - \frac{1}{4} T g_{ab} \right). \tag{16}$$

These equations imply that the trace-free part of the Einstein tensor (which equals the trace-free part of the Ricci tensor) is proportional to the trace-free part of the stress–energy tensor. This is equivalent to

$$G_{ab} = \kappa T_{ab} + \Lambda g_{ab}. \tag{17}$$

That is, for $\lambda = 1$, Rastall gravity is just ordinary Einstein gravity plus an arbitrary cosmological constant.

Formally this is the same as so-called "unimodular gravity" [27–32].[4] Note that for $\lambda = 1$ we have[5]

$$[T_R]_{ab} = T_{ab} - \frac{1}{4} T g_{ab}; \qquad T_R = 0. \tag{18}$$

So when reconstructing the physical stress–energy one simply has

$$T_{ab} = [T_R]_{ab} + \frac{1}{4} T g_{ab}; \qquad T_R = 0. \tag{19}$$

That is, from the physical stress–energy T_{ab} you can (uniquely) construct Rastall stress–energy $[T_R]_{ab}$. In contrast, from the stress–energy Rastall $[T_R]_{ab}$ you can reconstruct the physical stress–energy T_{ab}, up to an *a priori* unknown trace T. Consequently, even for $\lambda = 1$, Rastall gravity is a trivial rearrangement of the matter sector in Einstein gravity; as gravity there is absolutely nothing new.

5. Relation of Rastall to trace-free stress–energy

In terms of the usual stress–energy, let us define the trace-free stress–energy as

$$[T_{tf}]^{ab} = T^{ab} - \frac{1}{4} T\, g^{ab}. \tag{20}$$

While this trace-free stress–energy tensor certainly shows up in unimodular gravity [27–32], it has a much wider domain of applicability.

Naturally, this trace-free stress–energy, $[T_{tf}]^{ab}$, is not (generically) covariantly conserved, indeed we have $\nabla_b [T_{tf}]^{ab} = -\frac{1}{4} g^{ab} \nabla_b T$, but this covariant non-conservation is not at all a surprise, it is simply due to the way it has been defined.

Furthermore, since $T^{ab} - [T_{tf}]^{ab} = \frac{1}{4} T g^{ab}$, we can always rewrite the Rastall stress–energy of equation (12) as a simple linear interpolation between the physical and the trace-free stress–energy tensors:

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[2] Note the existence of an automatic implied consistency condition for Rastall stress–energy: $\nabla_{[c} \nabla^b [T_R]_{a]b} = 0$. This might at first glance look "deep"; unfortunately it is not "deep". Observe that one trivially has $\nabla_{[c} \nabla^b [T_R]_{a]b} = \frac{\lambda}{4} \nabla_{[c} \nabla_{a]} R = 0$.

[3] For traceless matter, such as electromagnetic stress–energy, the whole process trivializes, $[T_R]_{ab} \to T_{ab}$.

[4] Observe that "unimodular gravity" should more properly called "specified modulus gravity", meaning that $\det(g) \to \omega$, where ω is an externally specified and non-dynamical scalar density.

[5] Even for the special case $\lambda = 1$, there is still an automatic implied consistency condition for the Rastall stress–energy: $\nabla_{[c} \nabla^b [T_R]_{a]b} = 0$. This might again at first glance look "deep"; it isn't. We again trivially have $\nabla_{[c} \nabla^b [T_R]_{a]b} = \frac{1}{4} \nabla_{[c} \nabla_{a]} R = 0$.