Physics Letters B 781 (2018) 77–82

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On an algebraic structure of dimensionally reduced magical supergravity theories

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ARTICLE INFO

Article history:
Received 4 March 2018
Accepted 28 March 2018
Available online 29 March 2018
Editor: M. Cvetič

ABSTRACT

We study an algebraic structure of magical supergravities in three dimensions. We show that if the commutation relations among the generators of the quasi-conformal group in the super-Ehlers decomposition are in a particular form, then one can always find a parameterization of the group element in terms of various 3d bosonic fields that reproduces the 3d reduced Lagrangian of the corresponding magical supergravity. This provides a unified treatment of all the magical supergravity theories in finding explicit relations between the 3d dimensionally reduced Lagrangians and particular coset nonlinear sigma models. We also verify that the commutation relations of $E_{6(+2)}$, the quasi-conformal group for $\mathbb{A}=\mathbb{C}$, indeed satisfy this property, allowing the algebraic interpretation of the structure constants and scalar field functions as was done in the $F_{4(+4)}$ magical supergravity.

1. Introduction

One of the remarkable discoveries in the history of supergravity theories is that of the existence of five-dimensional exceptional supergravities associated with the Freudenthal–Tits magic square [1,2]. They are a special class of $D=5$ $N=2$ Einstein–Maxwell supergravity (having 8 supercharges) that exist in addition to the infinite series of the non-Jordan family of $D=5$ $N=2$ supergravity. There are four such theories associated with the four division algebras, whose scalar manifolds arising in the reductions to $D=4$ and 3 are coset manifolds of various (real forms of) exceptional and non-exceptional Lie groups, which, surprisingly enough, coincide with every entry of the table of the magic square that shows the symmetries of Jordan algebras (Table 1). Since the discovery many works have been done on these mysterious supergravity theories. An incomplete list includes [3–22]. (See also [23] for a review.)

More recently, a precise identification was made in [21] between the bosonic fields of the three-dimensional reduced smallest magical supergravity and the parameter functions of the coset space $F_{4(+4)}/(USp(6)\times SU(2))$, thereby the FFA couplings C_{IJK} of the magical supergravity were shown to be identifiable as particular structure constants of the quasi-conformal algebra of the relevant Jordan algebra. It was also clarified there that the scalar fields $\mathring{a}^{IJ}$ and $\mathring{a}_{IJ}$ are nothing but the metric of the reduced dimensions in a particular representation. Since the form of the dimensionally reduced Lagrangian is common to all magical supergravities, with the only differences being the range of the values of the indices of the vector and scalar fields, it was conjectured in [21] that such a Lie algebraic characterization of the coupling constants or the scalar fields also applies to other magical supergravities, not only to the smallest $\mathrm{J}_3^{\mathbb{R}}$ magical supergravity.

In this paper, we first show that if the commutation relations among the generators belonging to the respective irreducible components in the super-Ehlers decomposition are in a particular form, then one can always find a parameterization of the group element in terms of various 3d bosonic fields that reproduces the 3d reduced Lagrangian of the corresponding magical supergravity. This provides a unified treatment of all the magical supergravity theories in finding explicit relations between the 3d dimensionally reduced Lagrangians and particular coset nonlinear sigma models. This is done in section 2.

We then verify that the commutation relations of $E_{6(+2)}$, the quasi-conformal group for $\mathbb{A}=\mathbb{C}$, allows a decomposition whose generators indeed satisfy this property, which immediately proves that the 3d reduced $\mathbb{A}=\mathbb{C}$ magical supergravity consists of an $E_{6(+2)}/(SU(6)\times SU(2))$ nonlinear sigma model coupled to supergravity. This is done in section 3.

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https://doi.org/10.1016/j.physletb.2018.03.071
0370-2693/

Table 1
The magic square [1].

d	J				$\mathbb{A}$
	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$	
5 (compact)	$SO(3)$	$SU(3)$	$USp(6)$	F_4	$\mathbb{R}$
5 (non-compact)	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$	$\mathbb{C}$
4 (non-compact)	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$	$\mathbb{H}$
3 (non-compact)	$F_{4(+4)}$	$E_{6(+2)}$	$E_{7(-5)}$	$E_{8(-24)}$	$\mathbb{O}$

We should mention that $F_{4(+4)}$, the quasi-conformal group for $\mathbb{A}=\mathbb{R}$, has also been shown to have such a property [21]. We conjecture that the remaining quasi-conformal groups in the list ($E_{7(-5)}$ (for $\mathbb{A}=\mathbb{H}$) and $E_{8(-24)}$ (for $\mathbb{A}=\mathbb{O}$)) also possess such a special algebraic structure.

2. Explicit construction of 3d nonlinear sigma models using the super-Ehlers decomposition

The bosonic Lagrangian of a magical supergravity associated with the division algebra $\mathbb{A}$ is given by

$$\mathcal{L}=\frac{1}{2}E^{(5)}R^{(5)}-\frac{1}{4}E^{(5)}\overset{\circ}{a}_{IJ}F^{I}_{MN}F^{JMN}-\frac{1}{2}E^{(5)}s_{xy}(\partial_M\phi^x)(\partial^M\phi^y)+\frac{1}{6\sqrt{6}}C_{IJK}\epsilon^{MNPQR}F^I_{MN}F^J_{PQ}A^K_R. \quad (1)$$

$E^{(5)}$ is the determinant of the fünfbein, and $R^{(5)}$ is the scalar curvature in $D=5$. F^I_{MN} is the Ith Maxwell field strength $2\partial_{[\mu}A^I_{\nu]}$, where $I,J,\cdots=1,2,\ldots,n_{\mathbb{A}}+1$ with

$$n_{\mathbb{A}}=3(1+\dim\mathbb{A})-1. \quad (2)$$

$\overset{\circ}{a}_{IJ}$ and s_{xy} are functions of $n_{\mathbb{A}}$ scalar fields ϕ^x satisfying the relations $\overset{\circ}{a}_{IJ}=\overset{\circ}{a}_{JI}$ and $s_{xy}=s_{yx}$.

Following a standard procedure of dimensional reduction and field dualization, one finds a 3d dimensionally reduced dualized Lagrangian for all magical supergravities as [21]

$$\begin{aligned}\tilde{\mathcal{L}}=&\frac{1}{2}ER+\frac{1}{8}E\partial_\mu g^{mn}\partial^\mu g_{mn}-\frac{1}{2}Ee^{-2}\partial_\mu e\partial^\mu e\\&-\frac{1}{2}Es_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y)-\frac{1}{2}E\overset{\circ}{a}_{IJ}g^{mn}\partial_\mu A^I_m\partial^\mu A^J_n\\&-2Ee^{-2}\overset{\circ}{a}^{II'}\left(\partial_\mu\varphi_I-\frac{1}{\sqrt{6}}C_{IJK}\epsilon^{mn}\partial_\mu A^J_mA^K_n\right)\\&\times\left(\partial^\mu\varphi_{I'}-\frac{1}{\sqrt{6}}C_{I'J'K'}\epsilon^{m'n'}\partial^\mu A^{J'}_{m'}A^{K'}_{n'}\right)\\&-Ee^{-2}g^{mn}\left(\partial_\mu\psi_m+\partial_\mu A^I_m\varphi_I-A^I_m\partial_\mu\varphi_I\right.\\&\left.+\frac{2}{3\sqrt{6}}C_{IJK}\epsilon^{pq}\partial_\mu A^I_pA^J_qA^K_m\right)\\&\times\left(\partial^\mu\psi_n+\partial^\mu A^{I'}_n\varphi_{I'}-A^{I'}_n\partial^\mu\varphi_{I'}\right.\\&\left.+\frac{2}{3\sqrt{6}}C_{I'J'K'}\epsilon^{p'q'}\partial^\mu A^{I'}_{p'}A^{J'}_{q'}A^{K'}_n\right),\end{aligned} \quad (3)$$

where $\mu,\nu,\ldots$ are the 3d spacetime indices and $m,n,m',n',\ldots$ are the reduced two-dimensional indices. Note that this form is common to all the four magical supergravities; the only difference is the ranges the indices $x,y,\ldots$ and $I,J,\ldots$ run over.

Every magical supergravity contains in its 5d Lagrangian a nonlinear sigma model associated with the coset $\frac{\mathrm{Str}_0(J_3^{\mathbb{A}})}{\mathrm{Aut}(J_3^{\mathbb{A}})}$ [1,2]. When it is dimensionally reduced to three dimensions, the scalar coset is enlarged to $\frac{\mathrm{qConf}(J_3^{\mathbb{A}})}{\widetilde{\mathrm{M\ddot{o}}}(J_3^{\mathbb{A}})\times SU(2)}$, in which all the non-gravity bosonic degrees of freedom are contained. To show explicitly how the various terms arising through the dimensional reduction are gathered to form a single coset, it is convenient to decompose the Lie algebra of the quasi-conformal group $\mathrm{qConf}(J_3^{\mathbb{A}})$, which is the numerator group of the 3d coset, in terms of representations of the Lie algebra of the subgroup $SL(3,\mathbb{R})\times\mathrm{Str}_0(J_3^{\mathbb{A}})$, the latter of which is the numerator group of the 5d coset. The decomposition is always in the same form for all magical supergravities[18]:

$$\begin{aligned}&\mathrm{qConf}(J_3^{\mathbb{A}})\supset SL(3,\mathbb{R})\times\mathrm{Str}_0(J_3^{\mathbb{A}}),\\&\mathbf{adj}(\mathrm{qConf}(J_3^{\mathbb{A}}))=(\mathbf{8},\mathbf{1})\oplus(\mathbf{3},\overline{\mathbf{n}_{\mathbb{A}}+\mathbf{1}})\oplus(\bar{\mathbf{3}},\mathbf{n}_{\mathbb{A}}+\mathbf{1})\\&\qquad\oplus(\mathbf{1},\mathbf{adj}(\mathrm{Str}_0(J_3^{\mathbb{A}}))).\end{aligned} \quad (4)$$

We will show that if the generators of the Lie algebra of the quasi-conformal group $\mathrm{qConf}(J_3^{\mathbb{A}})$ take a particular form (5) as assumed below, then one can always reproduce the Lagrangian (3) as a coset nonlinear sigma model coupled to gravity.

Let the generators of the Lie algebra of the quasi-conformal group $\mathrm{qConf}(J_3^{\mathbb{A}})$ satisfy

$$\begin{aligned}&[\hat{E}^i{}_j,\ \hat{E}^k{}_l]=\delta^k_j\hat{E}^i{}_l-\delta^i_l\hat{E}^k{}_j,\\&[\hat{E}^i{}_j,E^{*k}_I]=\delta^k_jE^{*i}_I,\\&[\hat{E}^i{}_j,\ E^I_k]=-\delta^i_kE^I_j,\\&[T_{\tilde{i}},\ T_{\tilde{j}}]=f_{\tilde{i}\tilde{j}}{}^{\tilde{k}}T_{\tilde{k}},\\&[T_{\tilde{i}},E^{*k}_I]=\bar{t}_{\tilde{i}I}{}^JE^{*k}_J,\\&[T_{\tilde{i}},\ E^I_k]=t_{\tilde{i}}{}^I{}_JE^J_k,\\&[E^I_i,\ E^J_j]=C^{IJK}\epsilon_{ijk}E^{*k}_K,\\&[E^{*i}_I,\ E^{*j}_J]=-C_{IJK}\epsilon^{ijk}E^K_k,\\&[E^I_i,\ E^{*j}_J]=-2\delta^I_J\hat{E}^j_i+\delta^i_jD^I{}_J{}^{\tilde{i}}T_{\tilde{i}},\\&\text{otherwise}=0.\end{aligned} \quad (5)$$

ϵ^{ijk} and ϵ_{ijk} are completely antisymmetric tensors with $\epsilon_{123}=\epsilon^{123}=1$.

$\hat{E}^i{}_j$ $(i,j=1,\ldots,3)$ with $\hat{E}^1{}_1+\hat{E}^2{}_2+\hat{E}^3{}_3=0$ are the generators of $SL(3,\mathbb{R})$, the first irreducible component $(\mathbf{8},\mathbf{1})$ of (4). They are defined modulo $\hat{E}^1{}_1+\hat{E}^2{}_2+\hat{E}^3{}_3$, that is, $\hat{E}^i{}_j$ is an element of a quotient space of $GL(3,\mathbb{R})$ divided by the center generated by the overall $U(1)$ generator. $T_{\tilde{i}}$ $(\tilde{i}=1,\ldots,\dim\mathrm{Str}_0(J_3^{\mathbb{A}}))$ are the generators of $\mathrm{Str}_0(J_3^{\mathbb{A}})$ of the respective magical supergravity, which is the last irreducible component of (4). Finally, E^{*j}_J and E^I_i $(i,j=1,\ldots,3;\ I,J=1,\ldots,n_{\mathbb{A}}+1)$ are the generators of $(\mathbf{3},\overline{\mathbf{n}_{\mathbb{A}}+\mathbf{1}})$ and $(\bar{\mathbf{3}},\mathbf{n}_{\mathbb{A}}+\mathbf{1})$, respectively.

$f_{\tilde{i}\tilde{j}}{}^{\tilde{k}}$, $\bar{t}_{\tilde{i}I}{}^J$, $t_{\tilde{i}}{}^I{}_J$, C_{IJK}, C^{IJK} and $D^I{}_J{}^{\tilde{i}}$ are *real* structure constants; they are not all independent but are constrained by the Jacobi identities. The full set of constraints are

$$C^{IJK}=C^{JIK}=C^{JKI}=C^{KIJ}, \quad (6)$$

$$C_{IJK}=C_{JIK}=C_{JKI}=C_{KIJ}, \quad (7)$$

$$\bar{t}_{\tilde{i}I}{}^J=-t_{\tilde{i}}{}^J{}_I, \quad (8)$$

$$t_{\tilde{i}}{}^I{}_KC^{KJL}+t_{\tilde{i}}{}^J{}_KC^{KLI}+t_{\tilde{i}}{}^L{}_KC^{KIJ}=0, \quad (9)$$

$$t_{\tilde{i}}{}^K{}_IC_{KJL}+t_{\tilde{i}}{}^K{}_JC_{KLI}+t_{\tilde{i}}{}^K{}_LC_{KIJ}=0, \quad (10)$$