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The role of energy conditions in $f(R)$ cosmology

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ABSTRACT

Energy conditions can play an important role in defining the cosmological evolution. Specifically acceleration/deceleration of cosmic fluid, as well as the emergence of Big Rip singularities, can be related to the constraints imposed by the energy conditions. Here we discuss this issue for $f(R)$ gravity considering also conformal transformations. Cosmological solutions and equations of state can be classified according to energy conditions. The qualitative change of some energy conditions when transformation from the Jordan frame to the Einstein frame done is also observed.

1. Introduction

The observed cosmic acceleration [1–5] points out that a revision of the cosmological picture, based on the General Relativity (GR) and the standard model of particles, is needed. The puzzle can be addressed either introducing some form of dark energy or assuming modifications of GR. In other words, one can act either on the r.h.s. of the Einstein equations by introducing some new matter–energy fluid on the l.h.s. modifying or improving geometry. In this latter perspective, $f(R)$ gravity is the straightforward modification of GR where, instead of assuming the gravitational action strictly linear in the Ricci scalar R, one takes into account a general function of R. The paradigm is that the form of $f(R)$ can be fixed according to the cosmological and astrophysical observations ranging from local to cosmological scales [6–15].

Beside phenomenological approaches, first principles like energy conditions, causal structure and the classification of singularities can be considered to restrict the possible forms of $f(R)$ models [16–23]. In particular, energy conditions, originally formulated in Ref. [24] for GR, can play an important role to fix physically consistent $f(R)$ models [19]. In this debate, the role of conformal transformations is crucial because, also if the Jordan and Einstein frames are mathematically equivalent, the meaning of energy conditions can depend on the frame where they are formulated [25–28]. In particular, the effective pressure and effective energy definitions changes according to the frame [31,19–23,29,30] not only in $f(R)$ gravity but also in other alternative theories of gravity [32]. In general, it is important to define the role of further geometrical terms in the stress–energy tensor [33–36] and to recast the energy conditions accordingly. Conformal transformations and their physical meaning are crucial in the perspective of determining self-consistent energy conditions. For review, see [37–45].

In this paper, we are considering the role of energy conditions in of $f(R)$ cosmology. In particular, we discuss the conformal transformations of the $f(R)$ effective energy–momentum tensor. This issue is extremely relevant to address the attractive/repulsive behavior of $f(R)$ cosmological models in relation to the equation of state.

The paper is organized as follows. In Sec. 2, we consider the energy conditions in GR. Their definition for Extended Theories of Gravity (ETG) is taken into account in Sec. 3. The effective energy–momentum tensor, containing curvature terms, is discussed in Sec. 4. The relations of this generalized energy–momentum tensor to the cosmological equation of state are considered in Sec. 5.

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As an example of the above general results, we assume the case of power-law $f(R)$ gravity in Sec. 6. Conclusions are drawn in Sec. 7.

2. Energy conditions in General Relativity

Let us start from the Einstein field equations

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = \frac{\kappa^2}{2} T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is energy–momentum tensor of the matter fields. Such equations determine the causal and geodesic structure of space–time. The Einstein field equations can be written also as

$$R_{\mu\nu} = \frac{\kappa^2}{2}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right), \tag{2}$$

where the analog role of matter and geometry into dynamics is evident. Due to this aspect, we can deal with *geometrodynamics* after Wheeler [46]. Since such equations are addressing the causal (metric) and geodesic structure of the space–time, the energy–momentum tensor has to satisfy some conditions. We can take into account a timelike vector u^α normalized as $u^\alpha u_\alpha = -1$ for the signature $(-+++)$. It is the four-velocity of an observer in space–time, and an arbitrary, future-directed null vector k^α, i.e. $k^\alpha k_\alpha = 0$. The energy conditions are contractions of timelike or null vector fields with respect to the Einstein tensor and energy–momentum tensor coming from field Eqs. (1) or (2). We obtain four conditions [24,47] which are

- The **WEC** (WEC) which states that at each point of the space–time $\mathrm{p} \in \mathcal{M}$ the energy–momentum tensor satisfies the inequality

  $$T_{\mu\nu} u^\alpha u^\beta \geq 0, \tag{3}$$

  for any timelike vector $u \in T_{\mathrm{p}}\mathcal{M}$. If u^α is a four-velocity of an observer, then the quantity $T_{\mu\nu} u^\alpha u^\beta$ is the local energy density and the inequality (3) is equivalent to the assumption that the energy density of a given matter source, measured by an arbitrary observer, is non-negative. The canonical form of the energy–momentum tensor [24] can be written in the orthonormal basis as $T^{\mu\nu} = \mathrm{diag}(\rho, p_1, p_2, p_3)$ and then, one obtains

  $$\rho \geq 0, \quad \rho + p_i > 0, \quad i = 1, 2, 3. \tag{4}$$

  Following [35], it can be written as

  $$R_{\mu\nu} u^\mu u^\nu \geq -\frac{\kappa^2}{4}\left(\rho - \sum_{i=1}^{3} p_i\right). \tag{5}$$

- The **Null Energy Condition** (NEC) considers future-directed null vector k^μ

  $$T_{\mu\nu} k^\alpha k^\beta \geq 0, \tag{6}$$

  from which one gets $\rho + p_i \geq 0$.
- The **Dominant Energy Condition** (DEC) states that matter flows along timelike or null world lines. By contracting the energy–momentum tensor with an arbitrary, future-directed, timelike vector fields, the quantity $-T^\mu_\nu u^\nu$ becomes a future-directed, timelike or null vector field. It is called the matter momentum density that a given observer can measure. This means that, in any orthonormal basis, the energy dominates the other components of the energy–momentum tensor being $T^{00} \geq |T^{ij}|$:

  $$\rho \geq 0, \quad \rho \geq |p_i|. \tag{7}$$

- The **Strong Energy Condition** (SEC)

  $$\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) u^\mu u^\nu \geq 0 \tag{8}$$

  is a statement about the Ricci tensor:

  $$R_{\mu\nu} u^\mu u^\nu \geq 0, \tag{9}$$

  and together with the Raychaudhuri equation [48–51] gives that gravity has to be attractive.

All these considerations are related to standard matter which satisfies regular equations of state and is minimally coupled to the geometry. They can be generalized to other theories of gravity assuming that at least causal structure is preserved.

3. Energy conditions in Extended Theories of Gravity

Any alternative theory of gravity should be confronted with energy conditions which assign the fundamental causal and geodesic structure of space–time. In particular Extended Theories of Gravity (ETGs) [6–8], which are straightforward extensions of the Einstein gravity, can be recast in such a way to be dealt under the standard of energy conditions. As discussed in [35,36], the field equations of any ETG can be written in the form

$$g(\Psi^i)(G_{\mu\nu} + H_{\mu\nu}) = \frac{\kappa^2}{2} T_{\mu\nu}, \tag{10}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $g(\Psi^i)$ is a generalized coupling with the matter fields which contributes to the energy–momentum tensor $T_{\mu\nu}$. Ψ^i represents curvature invariants and/or gravitational fields which contributes to the dynamics. $H_{\mu\nu}$ is a geometric tensor term including all geometrical modifications given by the given ETG. General Relativity is recovered assuming $g(\Psi^i) = 1$ and $H_{\mu\nu} = 0$.

The contracted Bianchi identities and the covariant conservation of the energy–momentum tensor give the conservation law

$$\nabla_\alpha H^{\mu\nu} = -\frac{\kappa^2}{2g^2} T^{\mu\nu} \nabla_\alpha g, \tag{11}$$

which is zero if one deals with vacuum and the coupling g has a non-diverging value (i.e. $G_{\mu\nu} = -H_{\mu\nu}$). For energy conditions in ETGs, the combination of $G_{\mu\nu}$ and $H_{\mu\nu}$ is relevant while, in GR, one needs only the conditions for the Einstein tensor. Specifically, the extended SEC has the form

$$g(\Psi^i)\left(R_{\mu\nu} + H_{\mu\nu} - \frac{1}{2} g_{\mu\nu} H\right) u^\alpha u^\beta \geq 0, \tag{12}$$

from which one concludes that the condition $R_{\mu\nu} u^\mu u^\nu \geq 0$, valid for GR, does not guarantee the attractive nature of gravity. In other words, also in the case where SEC is valid, one can obtain repulsive gravity in ETGs, in particular in $f(R)$ gravity, as discussed in [52].

Physical quantities which are measured by an observer are the components of the energy–momentum tensor

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p h^{\alpha\beta} + \Pi^{\alpha\beta} + 2 q^{(\alpha} u^{\beta)}, \tag{13}$$

where $\rho = T_{\alpha\beta} u^\alpha u^\beta$ and $p = \frac{1}{3} T_{\alpha\beta} h^{\alpha\beta}$ are the energy-density and the isotropic pressure, respectively. $\Pi^{\alpha\beta} = \big(h^{\alpha\sigma} h^{\beta\gamma} -$