Physics Letters B 779 (2018) 353–357

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

A systematic and efficient method to compute multi-loop master integrals

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ARTICLE INFO

Article history:
Received 30 December 2017
Received in revised form 12 February 2018
Accepted 13 February 2018
Available online 19 February 2018
Editor: B. Grinstein

ABSTRACT

We propose a novel method to compute multi-loop master integrals by constructing and numerically solving a system of ordinary differential equations, with almost trivial boundary conditions. Thus it can be systematically applied to problems with arbitrary kinematic configurations. Numerical tests show that our method can not only achieve results with high precision, but also be much faster than the only existing systematic method sector decomposition. As a by product, we find a new strategy to compute scalar one-loop integrals without reducing them to master integrals.

1. Introduction

With the continuous improvement of statistics and experimental systematics at the Large Hadron Collider, the aim of testing the particle physics Standard Model and discovering new physics strongly demands theoretical predictions to also improve uncertainty to the same level. For many important processes, high order perturbative calculations are needed to this end. At the one-loop order, thanks to the improvement of traditional tensor reduction [1] and the development of unitarity-based reduction [2–4], one can efficiently express scattering amplitudes in terms of linear combinations of master integrals (MIs). As the computation of one-loop MIs is a solved problem [5–7], one-loop calculations can now be done automatically. Expressing multi-loop scattering amplitudes in terms of MIs is also possible using such as the integration-by-parts (IBP) reduction [8–13] or the unitarity-based multi-loop reduction [14–26]. Then, one of the main obstacles for multi-loop calculation is the computation of multi-loop MIs.

We take two recent studies in literature as examples to demonstrate how hard the multi-loop MIs computation is. One example is a two-loop calculation of pseudoscalar quarkonium inclusive decay [27], where the computational expense of MIs is about $\mathcal{O}(10^5)$ CPU core-hour. Another example is a calculation of four-loop nonplanar cusp anomalous dimension [28]. The reduction of amplitudes to MIs in this problem has been done much earlier in Ref. [29], yet the computation of these MIs is very challenging. The final numerical result obtained in Ref. [28] has uncertainty as large as 10%, which we believe is already the best precision that one can get with a tolerable computational expense.

Currently, the only method that can systematically compute any MI is the sector decomposition [30]. Unfortunately, this method is extremely time-consuming, besides that it is hard to achieve high precision. Mellin–Barnes representation [31] is another widely used method, yet it has difficulty to deal with non-planar diagrams, at least not in a systematic way (see Ref. [32] and references therein for recent progress). The differential equation (DE) method [33–36] is a powerful tool to compute multi-loop MIs, which bases on the fact that derivation of a MI with respect to its kinematic variables (including Mandelstam variables and internal masses) can be re-expressed as a linear combination of MIs using aforementioned reductions. For simple problems, DE method can give analytical results thanks to the introduction of canonical form [37–39]; while for complicated problems, one can solve DEs numerically to achieve results with high precision (see [40–43] and references therein). However, it needs input of boundary conditions (BCs) of MIs evaluated at another set of kinematic configurations. As there is no general rule to obtain BCs for arbitrary problems at present, one needs to find good BCs case by case, which makes it hard for DE method to be systematical. In practice, sector decomposition method is employed in Ref. [27], and both sector decomposition method and Mellin–Barnes representation method are employed in Ref. [28].

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https://doi.org/10.1016/j.physletb.2018.02.026

In this Letter, we develop a novel method to compute multi-loop MIs by constructing and solving a system of ordinary differential equations (ODEs). Advantages of our method are as follows: 1) Our BCs are fully massive vacuum integrals with a single mass scale, which are much simpler to compute and have been well studied in literature [44]. As a result, our method can be systematically applied to any complicated problem; 2) ODEs can be numerically solved efficiently to high precision, no matter how many mass scales are involved in the problem; 3) Computing MIs with complex kinematic variables is very easy in our method, while it could be hard for other methods (note that introducing imaginary part to kinematic variables is needed for many purposes, e.g., to describe particle decay or to study the S-matrix theory). Numerical tests show that our method can be much faster than the only existing systematic method sector decomposition. As a by product, we find a new strategy to compute scalar one-loop integrals in arbitrary spacetime dimensions without reducing them to MIs.

2. The method

Let us introduce a dimensionally regularized L-loop MI,

$$I(D;\{\nu_\alpha\};\eta)\equiv\int\prod_{i=1}^{L}\frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}}\prod_{\alpha=1}^{N}\frac{1}{(\mathcal{D}_\alpha+\mathrm{i}\eta)^{\nu_\alpha}}\,, \tag{1}$$

where D is the spacetime dimension, $\mathcal{D}_\alpha\equiv q_\alpha^2-m_\alpha^2$ are usual Feynman propagators, and q_α are linear combinations of loop momenta ℓ_i and external momenta p_i. The actual integral that we want to get is

$$I(D;\{\nu_\alpha\};0)\equiv\lim_{\eta\to 0^+}I(D;\{\nu_\alpha\};\eta), \tag{2}$$

with 0^+ defining the causality of Feynman amplitudes. In the following, we will suppress the dependence on D and $\{\nu_\alpha\}$ whenever it does not introduce any confusion.

We set up ODEs by differentiating MIs with respect to η and then re-expressing them in terms of MIs, which results in

$$\frac{\partial}{\partial\eta}\vec{I}(\eta)=A(\eta)\vec{I}(\eta)\,, \tag{3}$$

where $\vec{I}(\eta)$ is the vector of a complete set of m MIs and $A(\eta)$ is the $m\times m$ coefficient matrix. To obtain MIs at $\eta=0^+$, we solve the ODEs with BCs chosen at $\eta=\infty$. As we will show, BCs are simply vacuum integrals with equal masses, which can be computed rather easily. Considering also that numerically solving these ODEs is well-studied mathematical problem, our method provide a systematic and efficient way to compute multi-loop MIs to high precision.

3. Boundary conditions

Before studying BCs rigorously, let us explain the idea of choosing BCs at $\eta=\infty$. With a sufficiently large imaginary part in all denominators, we expect all kinematic variables to be negligible because they are finite. Thus we should be able to set both internal masses m_α and external momenta p_i to zero at the boundary, which results in simple vacuum integrals with equal masses. The only loophole in this argument is that, as loop momenta ℓ_i can be arbitrarily large, it is not obvious that $\ell_i\cdot p_j$ are negligible comparing with η even if $\eta\to\infty$. The loophole can be fixed by studying its Feynman parametric representation, and then our naïve expectation holds in general.

We assume $\nu_\alpha>0$ for all α in Eq. (1) to simplify our discussion, although our final conclusion is unchanged even without this condition. Then the Feynman parametric representation of Eq. (1) is given by

$$I(\eta)=(-1)^\nu\frac{\Gamma(\nu-LD/2)}{\prod_i\Gamma(\nu_i)}\int\prod_\alpha(x_\alpha^{\nu_\alpha-1}\mathrm{d}x_\alpha)\times\delta\left(1-\sum_j x_j\right)\frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U}-\mathrm{i}\eta)^{\nu-LD/2}}\,, \tag{4}$$

where $\mathcal{U}$ and $\mathcal{F}$ are so-called graph polynomials that can be related to the spanning 1-tree and 2-tree of the original Feynman diagram, respectively (see e.g. Ref. [45]), and ν is short for $\sum_\alpha\nu_\alpha$. All kinematic variables are incorporated in $\mathcal{F}$, leaving $\mathcal{U}$ depending only on Feynman parameters.

An important observation is that $|\mathcal{F}/\mathcal{U}|$ is bounded in the open interval of Feynman parameter space. To show this, we express $\mathcal{F}=\sum_i\mathcal{F}_i$ and $\mathcal{U}=\sum_i\mathcal{U}_i$, where $\mathcal{F}_i$ and $\mathcal{U}_i$ are monomials in Feynman parameters. By definition, a 2-tree can be generated by a 1-tree, i.e. there exists a pair of indexes j and k so that $\mathcal{F}_i=t_i\mathcal{U}_jx_k$, where t_i is the kinematic part of $\mathcal{F}_i$. We then have $|\mathcal{F}_i|<|t_i||\mathcal{U}_i|<|t_i||\mathcal{U}|$ and $|\mathcal{F}|<\sum_i|t_i||\mathcal{U}|$, where we have used the fact that $\mathcal{U}_i$ are positive definite in the open interval. As $\sum_i|t_i|$ is finite, we conclude that $|\mathcal{F}/\mathcal{U}|$ is bounded.

Because $|\mathcal{F}/\mathcal{U}|$ is bounded, $\mathcal{F}/\mathcal{U}$ in the denominator of Eq. (4) can be neglected as $\eta\to\infty$. This effectively sets all kinematic variables to zero in the original integral, because $\mathcal{F}$ includes all kinematic variables. The result is a fully massive vacuum integral $I^{\mathrm{bub}}(\eta)$ which shares the same internal topology as the original integral. Because this is a single scale integral, the η dependence can be factorized out, which results in a relation

$$I(\eta)=\eta^{LD/2-\nu}\Big[I^{\mathrm{bub}}(1)+\mathcal{O}(\eta^{-1})\Big], \tag{5}$$

where $I^{\mathrm{bub}}(1)$ can be interpreted as a vacuum integral with equal internal squared masses $m^2=-\mathrm{i}$. It is worth mentioning that the object $J(\eta)\equiv\eta^{\nu-LD/2}I(\eta)$ is analytic near $\eta=\infty$ based on the above discussion.

To compute $I^{\mathrm{bub}}(1)$, we again reduce it to linear combination of corresponding vacuum MIs, diagrams of which up to 3 loops are shown in Fig. 1. Computation of these vacuum MIs is well studied, with analytical results available up to 3 loops [46–48] (see [49] and references therein for some pioneering works) and numerical results available up to 5 loops [44,50,51]. We therefore conclude that the computation of BCs in our method is a solved problem.

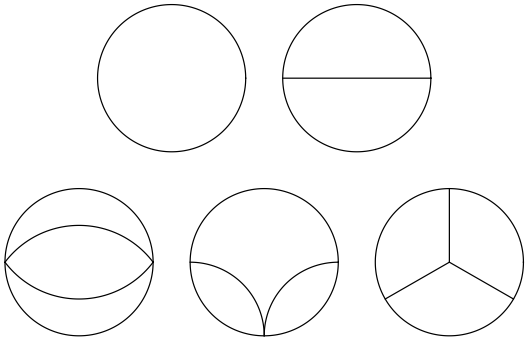

Fig. 1. Diagrams of nonfactorizable vacuum master integrals up to 3 loops.

4. Solving ODEs numerically

Knowing BCs, solving the ODEs numerically to obtain MIs at $\eta=0^+$ is a well-studied mathematical problem. The solution can be obtained efficiently to high precision.