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Resonant kink–antikink scattering through quasinormal modes

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ABSTRACT

We investigate the role that quasinormal modes can play in kink–antikink collisions, via an example based on a deformation of the ϕ^4 model. We find that narrow quasinormal modes can store energy during collision processes and later return it to the translational degrees of freedom. Quasinormal modes also decay, which leads to energy leakage, causing a closing of resonance windows and an increase of the critical velocity. We observe similar phenomena in an effective model, a small modification of the collective-coordinate approach to the ϕ^4 model.

1. Introduction

It is well-known that kinks in nonintegrable models such as the ϕ^4 theory can interact in a complicated way. One of the most interesting features is the existence of a resonance structure in kink–antikink collisions [1–3]. During the initial impact (or 'bounce'), oscillational modes can be excited, storing energy which on recollision can be given back to the translational modes of the kink and antikink. If the initial velocities are right, a significant fraction of the energy is returned, and kink–antikink pair can reseparate after one or more further bounces, albeit with the loss of some energy to radiation For other initial velocities less energy is returned, and the kink–antikink pair annihilate, leading to a 'fractal' structure of nested escape windows [4,5].

Such features were reported in many different models, including the double sine Gordon model [3], a coupled nonlinear Schrödinger equation [6], and a two-component ϕ^4 model [7–9]. A collision of a kink with a suitable impurity [10–12] or with a nontrivial boundary [13,14] can also lead to resonant behaviour and a fractal structure. Furthermore, a boundary collision can induce boundary decay with the associated creation of an extra kink or antikink, resulting in a secondary resonant structure [13].

For a long time it was thought that the existence of an oscillational mode of the kink was a necessary condition for the formation of a resonant structure. More recently, it was shown that even in models such as the ϕ^6 theory, where kinks have no internal oscillational modes, a fractal structure can still be observed, with modes trapped in the interval between the kink and antikink standing in for the localised modes [15]. This new mechanism can be expected to lead to a fractal in many cases of asymmetric kinks in models with different masses of small perturbations around different vacua [16–20]. Some efforts have also been made to reproduce the resonant structure of the ϕ^4 theory in more realistic situations, such as graphene ribbons [21].

In this paper we exhibit yet another mechanism which can lead to resonant scattering. Energy can also be stored in narrow resonance modes, which in order to avoid confusion with the resonant structure will be called quasinormal modes throughout this paper. Quasinormal modes (QNM) are especially long-lived states which are in some senses similar to oscillational modes, though they satisfy purely outgoing boundary conditions and hence are not normalisable. They decay exponentially, losing energy due to their radiative tails.

1.1. Quasinormal modes

Quasinormal modes play important roles both in quantum and classical physics. They satisfy purely outgoing wave boundary conditions, breaking the hermicity of the Hamiltonian. As a result, in quantum physics, they have complex energies $E = E_r + i\Gamma$. The imaginary part Γ is responsible for exponential decay of the state. One of the earliest applications of this idea was in the explanation of radioactivity: a nucleus forms an effective potential barrier which almost traps a particle, but which vanishes at larger distances, allowing the particle to tunnel through it. QNMs can be also seen as peaks of crosssections in scattering processes.

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Quasinormal modes are also often important in the classical evolution of dynamical systems, and indeed that is the context where they were first discussed [22]. The long-time dynamics are governed by the poles of the Green's function [23–25], with the position of the pole determining the nature of the mode. Poles corresponding to real frequencies are normal oscillational modes which in the linear approximation last infinitely long. Poles corresponding to imaginary frequencies are unstable modes which grow exponentially fast. The poles for complex frequencies describe QNM, which represent decaying oscillations [23]. For massive fields so called threshold modes decaying according to some power law can also dominate the long time dynamics [24]. One of the most surprising features of QNM is that their dynamics can have nonlinear tails which start to dominate when the mode decays below a certain amplitude [26].

The most notable current applications of QNMs are for signals of merging black holes. From gravitational wave measurements they can for example be used to find the masses of the colliding black holes [27–29].

2. The model

2.1. Recalling the ϕ^4 model

In the following we limit our considerations to 1+1 dimensional theories of a single scalar field:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \tag{1}$$

The first example of the resonant scattering mechanism was found for the ϕ^4 theory, with the field theory (scalar) potential

$$U(\phi) = \frac{1}{2}(\phi^2 - 1)^2 \equiv W. \tag{2}$$

The two vacuum configurations $\phi(x) = \phi_\pm = \pm 1$ break the $\mathbb{Z}_2$ symmetry of the model; kinks and antikinks are stationary solutions which interpolate between these vacua. The static kink solution can be found from the BPS equation

$$\phi'_K(x) = \sqrt{2U(x)}, \tag{3}$$

with a solution

$$\phi_{K,\bar{K}}(x) = \pm\tanh(x). \tag{4}$$

Small perturbations around the kink $\phi(x,t) = \phi_K(x) + e^{i\omega t}\eta(x)$ satisfy the linearised equation

$$-\eta'' + V(x)\eta = \omega^2\eta, \qquad V(x) = U''(\phi(x))\Big|_{\phi=\phi_K} \tag{5}$$

which has the form of a Schrödinger equation with a 'potential' for the linearised fluctuations given by $V(x)$.[1] In this particular case this is the famous Pöschl–Teller potential

$$V(x) = 4 - \frac{6}{\cosh^2(x)} \tag{6}$$

and it supports two bound states, with frequencies $\omega_0 = 0$ and $\omega_d = \sqrt{3}$. The first is the translational mode of the kink, while the second is referred to as the oscillational mode. The existence of this oscillational mode leads to the resonance windows during kink–antikink collisions. Some of the initial kinetic energy is stored in the oscillational mode, which for appropriate (resonant) initial conditions can be given back to the translational degrees of freedom in a subsequent recollision. The kink and antikink can bounce multiple times and either separate or end their existence as an oscillon. The resonant structure is very complicated, exhibiting fractal-like properties. The model has been studied extensively using both numerical and analytical methods. An effective model was introduced [1,2] which later was used in many variants and approximations [4] and reproduced reasonably well both the fractal structure, and the critical velocity above which no multibounce windows are observed. However, it is important to note that the initial effective model contained some errors, which were corrected in [30].

2.2. Designing the model

Our aim is to study the influence of QNM on collision scenarios similar to those known in the literature. The kink of the standard ϕ^4 model, defined above, does not have QNM in its spectrum of small perturbations. This is a rare feature, in this case a consequence of the reflectionlessness nature of the linearised potential for fluctuations about the ϕ^4 kink. Our strategy will be to modify the field theory potential W so as to turn the oscillational mode about the kink into a quasinormal mode. The linearised potential for the ϕ^4 kink tends to the asymptotic value $V(|x| \to \infty) = 4$, meaning that waves with frequencies below 2 cannot propagate. However if at some distance from the kink this potential would decrease further, changing its asymptotic to $V(|x| \to \infty) = m^2 < 4$, waves with frequencies below 2 but above m would become able to propagate. In particular if $m < \sqrt{3}$ the oscillational mode could tunnel through the barrier and would become a quasinormal mode. We will use of this observation to design a model for which the linearised potential for fluctuations about a static kink is very similar to that of the ϕ^4 model and yet its height decreases as $|x| \to \infty$.

It is worth mentioning that having the linearised potential $V(x)$ one can in principle reconstruct the field theory potential $U(\phi)$ [31,32]. However, except some rare cases, the procedure gives a very complicated potential which only can be found numerically, so we will not adopt this approach. Instead, we look for a field theory potential which is very similar to the ϕ^4 potential, $U \approx W$, when the field is far away from either vacuum. But when the field approaches one or other vacuum, which will happen far from the kink, the behaviour of the potential should change. Recall that $V(x) = U''(\phi(x))$. For $\phi = \pm 1$ the linearised potential is equal to m^2, which is the squared mass of the scalar field. For the ϕ^4 theory with our normalisations, this mass is equal to 2. The second feature which we want for our field theory potential is that its second derivative around the vacuum $\phi = \pm 1$ would be $m^2 < 3$ to allow the oscillational mode to tunnel through the barrier and to become the QNM.

We have found one such family of field theory potentials to be

$$U(\phi, \epsilon) = W + \frac{m^2 - 4}{4}\frac{\epsilon W}{W + \epsilon}, \tag{7}$$

which for $\epsilon = 0$ restores the standard ϕ^4 potential. For $\epsilon > 0$ the potential has a shape close to ϕ^4, but near vacua (where $W \lesssim \epsilon$) it behaves as a field with mass m. Unless stated otherwise throughout the paper $m = 1$. Some examples of this potential for different values of ϵ are shown in Fig. 1.

The ϕ^4 kink approaches its vacuum as

$$\phi_{\phi^4}(x) = \tanh x \approx 1 - 2e^{-2x} \tag{8}$$

For small values of ϵ, the additional term in the potential becomes important when $W \approx \epsilon$, which is for

[1] To avoid confusion with the field theory potential U, we will sometimes refer to V as the linearised potential.