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Multicritical points of the $O(N)$ scalar theory in $2 < d < 4$ for large N

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ABSTRACT

We solve analytically the renormalization-group equation for the potential of the $O(N)$-symmetric scalar theory in the large-N limit and in dimensions $2 < d < 4$, in order to look for nonperturbative fixed points that were found numerically in a recent study. We find new real solutions with singularities in the higher derivatives of the potential at its minimum, and complex solutions with branch cuts along the negative real axis.

1. Introduction

The $O(N)$-symmetric scalar theories have served for decades as the testing ground of techniques developed for the investigation of the critical behaviour of field theories and statistical models. It comes, therefore, as a surprise that a recent study [1] has found that their phase structure may be much more complicated that what had been found previously. In particular, it is suggested that, in dimensions $2 < d < 4$, several nonperturbative fixed points exist, which had not been identified until now. The large-N limit [2–7] offers the possibility to identify such fixed points analytically, without resorting to perturbation theory. We shall consider the theory in this limit through the Wilsonian approach to the renormalization group (RG) [8]. Its various realizations [9–13] give consistent descriptions of the fixed-point structure of the three-dimensional theory [14], in agreement with known results for the Wilson–Fisher (WF) fixed point [15] and the Bardeen–Moshe–Bander (BMB) endpoint of the line of tricritical fixed points [16–18].

We shall employ the formalism of ref. [11], leading to the exact Wetterich equation for the functional RG flow of the action. For $N \to \infty$ the anomalous dimension of the field vanishes and higher-derivative terms in the action are expected to play a minor role. This implies that the derivative expansion of the action [19–21] can be truncated at the lowest order, resulting in the local potential approximation (LPA) [9,13,14,22]. The resulting evolution equation for the potential is exact in the sense explained in ref. [14]. It has been analysed in refs. [23,24] in three dimensions. In this work, we extend the analysis over the range $2 < d < 4$, in an attempt to identify new fixed points.

2. Evolution equation for the potential

We consider the theory of an N-component scalar field ϕ^a with $O(N)$ symmetry in d dimensions. We are interested in the functional RG evolution of the action as a function of a sharp infrared cutoff k. We work within the LPA approximation, neglecting the anomalous dimension of the field and higher-derivative terms in the action. We define $\rho = \frac{1}{2}\phi^a\phi_a$, $a = 1...N$, as well as the rescaled field $\tilde{\rho} = k^{2-d}\rho$. We denote derivatives with respect to $\tilde{\rho}$ with primes. We focus on the potential $U_k(\rho)$ and its dimensionless version $u_k(\tilde{\rho}) = k^{-d}U_k(\rho)$. In the large-$N$ limit and for a sharp cutoff, the evolution equation for the potential can be written as [23]

$$\frac{\partial u'}{\partial t} = -2u' + (d-2)\tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} - \frac{NC_d}{1+u'}\frac{\partial u'}{\partial \tilde{\rho}}, \tag{1}$$

with $t = \ln(k/\Lambda)$ and $C_d^{-1} = 2^d\pi^{d/2}\Gamma(d/2)$. This equation can be considered as exact, as explained in ref. [14]. The crucial assumption is that, for $N \to \infty$, the contribution from the radial mode is negligible compared to the contribution from the N Goldstone modes.

The most general solution of eq. (1) can be derived with the method of characteristics, generalizing the results of ref. [23]. It is given by the implicit relation

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$$\tilde{\rho} - \frac{NC_d}{d-2}\, {}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -u'\right) = e^{(2-d)t}\, G\left(u' e^{2t}\right) - \frac{NC_d}{d-2} e^{(2-d)t}\, {}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -u' e^{2t}\right), \quad (2)$$

with ${}_2F_1(a, b, c, z)$ a hypergeometric function. The function G is determined by the initial condition, which is given by the form of the potential at the microscopic scale $k = \Lambda$, i.e. $u'_\Lambda(\tilde{\rho}) = \Lambda^{-2} U'_\Lambda(\rho)$. G is determined by inverting this relation and solving for $\tilde{\rho}$ in terms of u', so that $G(u') = \tilde{\rho}(u')|_{t=0}$. The effective action is determined by the evolution from $k = \Lambda$ to $k = 0$.

We are interested in determining possible fixed points arising in the context of the general solution (2). Infrared fixed points are approached for $k \to 0$ or $t \to -\infty$. For finite u', the last argument of the hypergeometric function in the rhs of eq. (2) vanishes in this limit. Using the expansion

$${}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -z\right) = 1 + \frac{d-2}{4-d} z - \frac{d-2}{6-d} z^2 + \mathcal{O}(z^3) \quad (3)$$

we obtain

$$\tilde{\rho} - \frac{NC_d}{d-2}\, {}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -u'\right) = e^{(2-d)t} \left(G\left(u' e^{2t}\right) - \frac{NC_d}{d-2} \right). \quad (4)$$

The t-dependence in the rhs must be eliminated for a fixed-point solution to exist. This can be achieved for appropriate functions G. For example, we may assume that the initial condition for the potential at $k = \Lambda$ or $t = 0$ is $u_\Lambda(\tilde{\rho}) = \lambda_\Lambda (\tilde{\rho} - \kappa_\Lambda)^2/2$, so that $G(z) = \kappa_\Lambda + z/\lambda_\Lambda$. Through the unique fine tuning $\kappa_\Lambda = NC_d/(d-2)$ the rhs vanishes for $t \to -\infty$. The scale-independent solution, given by the implicit relation

$$\tilde{\rho} - \frac{NC_d}{d-2}\, {}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -u'_*\right) = 0, \quad (5)$$

describes the Wilson–Fisher fixed point.

Near the minimum of the potential, where $u'_* \simeq 0$, we have

$$\tilde{\rho} - \frac{NC_d}{d-2} - \frac{NC_d}{4-d} u'_* + \frac{NC_d}{6-d} (u'_*)^2 + \mathcal{O}\left((u'_*)^3\right) = 0. \quad (6)$$

From this relation we can deduce that the minimum is located at $\tilde{\rho} = NC_d/(d-2) \equiv \kappa_*$, while the lowest derivatives of the potential at this point are $u''_*(\kappa_*) = (4-d)\,(NC_d)^{-1}$, $u'''_*(\kappa_*) = 2(4-d)^3/(6-d)\,(NC_d)^{-2}$. For large u'_*, we can use the expansion

$${}_2F_1\left(1, 1-\frac{d}{2}, 2-\frac{d}{2}, -z\right) = \Gamma\left(2-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right) z^{\frac{d}{2}-1} + \mathcal{O}\left(z^{\frac{d}{2}-2}\right) \quad (7)$$

in order to obtain the asymptotic form of the potential: $u_*(\tilde{\rho}) \sim \tilde{\rho}^{d/(d-2)}$. This result is consistent with the expected critical exponent $\delta = (d+2)/(d-2)$ for vanishing anomalous dimension. Finally, we note that the hypergeometric function has a pole at $z = -1$. This implies that, in the regions of negative u', the unrescaled potential $U_k \tilde{\rho})$ becomes flat, with its curvature scaling as $-k^2$ for $k \to 0$ [25].

We are interested in the existence of additional fixed points. In $d = 3$ it is known that, apart from the Wilson–Fisher fixed point, a line of tricritical fixed points exists, terminating at the BMB fixed point [16–18]. In the following section we describe the flows between these fixed points in terms of the potential, in order to obtain useful intuition for the investigation of the case of general d. Our analysis extends the picture of refs. [23,24] away from the fixed points.

3. $d = 3$

For $d = 3$, the solution (2) reproduces the one presented in ref. [23], through use of the identity

$${}_2F_1\left(1, -\frac{1}{2}, \frac{1}{2}, -z\right) = \sqrt{z}\, \arctan(\sqrt{z}) + 1 \qquad z > 0$$

$${}_2F_1\left(1, -\frac{1}{2}, \frac{1}{2}, -z\right) = \frac{1}{2}\sqrt{-z}\, \ln\left(\frac{1-\sqrt{-z}}{1+\sqrt{-z}}\right) + 1 \qquad z < 0. \quad (8)$$

In order to deduce the phase diagram of the three-dimensional theory, we consider a bare potential of the form

$$u'_\Lambda(\tilde{\rho}) = \lambda_\Lambda (\tilde{\rho} - \kappa_\Lambda) + \nu_\Lambda (\tilde{\rho} - \kappa_\Lambda)^2. \quad (9)$$

The solution (2) can be written as

$$\tilde{\rho} - \kappa_*\, {}_2F_1\left(1, -\frac{1}{2}, \frac{1}{2}, -u'\right) = e^{-t}\left[G\left(u' e^{2t}\right) - \kappa_*\, {}_2F_1\left(1, -\frac{1}{2}, \frac{1}{2}, -u' e^{2t}\right)\right], \quad (10)$$

with $\kappa_* = N/(4\pi^2)$. The function $G(z)$ is obtained by solving eq. (9) for $\tilde{\rho}$ as a function of u'_Λ. It is given by

$$G(z) = \kappa_\Lambda + \frac{1}{2\nu_\Lambda}\left(-\lambda_\Lambda \pm \sqrt{\lambda_\Lambda^2 + 4\nu_\Lambda z}\right), \quad (11)$$

with the two branches covering different ranges of $\tilde{\rho}$.

Let us impose the fine tuning $\kappa_\Lambda = \kappa_*$, which puts the theory on the critical surface. For $\lambda_\Lambda \neq 0$, we have $G(u' e^{2t}) \simeq \kappa_* + u' e^{2t}/\lambda_\Lambda$ for $t \to -\infty$. We also have ${}_2F_1\left(1, -1/2, 1/2, -u' e^{2t}\right) \simeq 1 + u' e^{2t}$. As a result, the rhs of eq. (10) vanishes in this limit. The evolution leads to the Wilson–Fisher fixed point discussed in the previous section. The additional fine tuning $\lambda_\Lambda = 0$ results in a different situation. For $t \to -\infty$ the rhs of eq. (10) becomes t-independent and we obtain

$$\tilde{\rho} - \kappa_*\, {}_2F_1\left(1, -\frac{1}{2}, \frac{1}{2}, -u'_*\right) = \pm \frac{1}{\sqrt{\nu_\Lambda}} \sqrt{u'_*}. \quad (12)$$

A whole line of tricritical fixed points can be approached, parametrized by ν_Λ [24]. Each of them is expected to be unstable towards the Wilson–Fisher fixed point.

The relative stability of the fixed points can be checked explicitly by considering the full solution (2). In Fig. 1 we depict the evolution of the potential, as predicted by this expression, for $\lambda_\Lambda = 10^{-7}$ and $\nu_\Lambda = 0.3$. We have set $NC_3 = 1$ through a redefinition of $\tilde{\rho}$ and u'. We have indicated by UV the initial form of the potential at $k = \Lambda$ and with IR its form for $k \to 0$. The continuous lines depict the potential at various values of t, with step equal to -1, during its initial approach to the tricritical fixed point (TP). The dashed lines depict its subsequent evolution towards the Wilson–Fisher fixed point (WF).

We shall not analyse in detail the tricritical line, as this has been done elsewhere [16–18,24]. We note that it connects the Gaussian fixed point, for $\nu_\Lambda = 0$, with a point approached for a value of ν_Λ for which the solution of eq. (12) diverges at the origin. This endpoint of the tricritical line is the BMB fixed point [16].