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The role of Weyl symmetry in hydrodynamics

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ABSTRACT

This article is dedicated to the analysis of Weyl symmetry in the context of relativistic hydrodynamics. Here is discussed how this symmetry is properly implemented using the prescription of minimal coupling: $\partial \to \partial + \omega\mathcal{A}$. It is shown that this prescription has no problem to deal with curvature since it gives the correct expressions for the commutator of covariant derivatives.

In hydrodynamics, Weyl gauge connection emerges from the degrees of freedom of the fluid: it is a combination of the expansion and entropy gradient. The remaining degrees of freedom, shear, vorticity and the metric tensor, are see in this context as charged fields under the Weyl gauge connection. The gauge nature of the connection provides natural dynamics to it via equations of motion analogous to the Maxwell equations for electromagnetism. As a consequence, a charge for the Weyl connection is defined and the notion of local charge is analyzed generating the conservation law for the Weyl charge.

1. Introduction

The holographic calculation of the η/s ratio in the $\mathcal{N}=4$ SYM plasma and the production of the Quark–Gluon plasma at RICH and LHC reveled the relativistic nature of this fluid and the scaling symmetry of the system [1,2]. The trace of Weyl symmetry comes either from the experimental bound on bulk viscosity and from the similarity of the η/s calculation in AdS/CFT to the experimental data. This phenomenology calls attention to the importance of the dynamics of a relativistic fluid with Weyl invariance. The formulation of the relativistic fluid dynamics is performed in a gradient expansion approach [3–8]. The elastic and friction properties of the fluid are described as perturbations of the ideal fluid flow and the dispersion relation is expressed in a power series $\omega(k) = \omega_0 + \delta_1 k + \delta_2 k^2 + \ldots$ Perturbations are considered at the level of energy-momentum tensor, starting with ideal fluid

$$T^{ab}_{ideal} = \epsilon u^a u^b + p\Delta^{ab}, \tag{1}$$

with ϵ the energy density, p the pressure, u^a a normalized velocity field ($g_{ab}u^a u^b = -1$) and $\Delta^{ab} = g^{ab} + u^a u^b$ is the transverse projector. The signature of the metric is $diag(g) = (-, +, \ldots, +)$. The gradient expansion of energy momentum-tensor is

$$T^{ab} = (\epsilon + E)u^a u^b + (p + P)\Delta^{ab} + q^{(a}u^{b)} + \pi^{\langle ab\rangle}, \tag{2}$$

that includes scalar (E, P), vector (q^a) and tensor $\pi^{\langle ab\rangle}$ perturbations of the ideal fluid. The symmetric, transverse and traceless part of a contra-variant rank 2 tensor is $\mathcal{T}^{\langle ab\rangle} = \frac{1}{2}\Delta^a{}_c\Delta^b{}_d(\mathcal{T}^{cd} + \mathcal{T}^{dc}) - \frac{1}{d-1}\Delta^{ab}\Delta_{cd}\mathcal{T}^{cd}$. These perturbations arise from the complete set of hydrodynamical degrees of freedom: the fluid entropy $\nabla^a_\perp \ln s$, the velocity $\nabla_{\perp a}u^b$ and the torsion free connection $\Gamma^c{}_{ab} = \frac{1}{2}g^{ck}(\partial_a g_{bk} + \partial_b g_{ka} - \partial_k g_{ab})$ presents in the covariant derivatives and in the curvature tensors. Velocity gradient is separated by symmetry, we have expansion $\Theta = \nabla_{\perp a}u^a$, shear $\sigma^{ab} \equiv 2\nabla^{\langle a}_\perp u^{b\rangle}$ and vorticity $\Omega^{ab} \equiv \nabla^{[a}_\perp u^{b]}$.

The ideal fluid equations together with the thermodynamical equations fixes that $\nabla^a_\perp \ln s = -(d-1)u^c\nabla_c u^a$, $u^a\nabla_a \ln s = -\Theta$, reducing the effective degrees of freedom. Gradient expansion is them completely fixed by shear, vorticity, expansion, entropy gradient $\nabla^a_\perp \ln s$, Riemann tensor and their derivatives. The number of derivatives in each structure is the order of gradient expansion where it appears and for each structure in this expansion there is one transport coefficient associated. Weyl symmetry constrains the dynamics of the fluid, reducing the number of transport coefficients in all orders. When this symmetry takes place expansion and entropy gradient are no longer allowed by symmetry in the gradient expansion, instead they combine forming the gauge con-

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nection. It is shown that minimal coupling correctly realizes Weyl invariance even when it comes to curvature structures. The gauge structure of Weyl symmetry is explored in the context of relativistic hydrodynamics, revealing that a constraint to its dynamics due to a conservation law associated with the Weyl gauge charge.

This article is organized as follows: section 2 is dedicated to analyze Weyl symmetry in a general scenario and the minimal coupling prescription is established, in section 3 is discussed the consequences of this symmetry in a hydrodynamical system, section 4 deals with the notion of local charge conservation for the Weyl gauge field and in section 5 are made some final comments.

2. Minimal coupling prescription

To require Weyl symmetry we impose that the system is invariant under the local scaling transformation

$$g^{ab} \to e^{-2\phi(x)} g^{ab}, \quad g_{ab} \to e^{2\phi(x)} g_{ab}. \tag{3}$$

We say that g^{ab} has scaling weight 2 while g_{ab} has scaling weight -2. The local scaling of the metric requires a scaling of the velocity field $u^a \to e^{-\phi(x)} u^a$. Weyl symmetry is a local scaling invariance which in some cases is equivalent to conformal symmetry [9,10]. For an hydrodynamic system to be invariant under such a transformation its energy-momentum tensor should transform as a tensorial density with scaling weight $\omega_T = d+2$:

$$T^{ab} \to e^{-(d+2)\phi} T^{ab}. \tag{4}$$

Consequently, in the gradient expansion only Weyl covariant perturbations are allowed.

A non-trivial role of the Weyl scaling is that it changes the Christoffel connection in a inhomogeneous way:

$$\Gamma^c{}_{ab} \to \Gamma^c{}_{ab} + \delta^c{}_a \nabla_b \phi + \delta^c{}_b \nabla_a \phi - g_{ab} \nabla^c \phi.$$

As a consequence the velocity gradients and the curvature tensor also transforms inhomogeneously under Weyl transformations. To repair the inhomogeneous terms of the hydrodynamical degrees of freedom it was introduced in [11] a Weyl covariant derivative that acts in the fields preserving their character of tensor density. Take a vector field ζ^b with scaling weight ω_ζ, $\zeta^b \to e^{-\omega_\zeta \phi} \zeta^b$, its Weyl covariant derivative is

$$\mathcal{D}_a \zeta^b = \nabla_a \zeta^b + \omega_\zeta \mathcal{A}_a \zeta^b + (g_{ac} \mathcal{A}^b - \delta^b{}_a \mathcal{A}_c - \delta^b{}_c \mathcal{A}_a) \zeta^c, \tag{5}$$

where $\mathcal{A}$ is the Weyl connection and transform as $\mathcal{A}_a \to \mathcal{A}_a + \partial_a \phi$ under Weyl transformations. The mathematical properties of this connection are discussed in [12]. This structure of the Weyl covariant derivative can be organized in a subtle way. It is useful to look at Weyl covariant derivative using the explicit form of the Christoffel connection:

$$\mathcal{D}_a \zeta^b = \partial_a \zeta^b + \omega_\zeta \mathcal{A}_a \zeta^b + \frac{1}{2} g^{bk} (\partial_c g_{ak} + \partial_a g_{kc} - \partial_k g_{ca}) \zeta^c + (g_{ac} \mathcal{A}^b - \delta^b{}_a \mathcal{A}_c - \delta^b{}_c \mathcal{A}_a) \zeta^c. \tag{6}$$

We can write the first two terms in the nice composition $(\partial_a + \omega_\zeta A_a)\zeta^b$. Moreover, we can proceed in the same way and identify the terms of the Christoffel connection with the terms of the Weyl gauge connection in the simple form

$$\mathcal{D}_a \zeta^b = (\partial_a + \omega_\zeta \mathcal{A}_a) \zeta^b + \frac{1}{2} g^{bk} [(\partial_c + \omega_g \mathcal{A}_c) g_{ak} + (\partial_a + \omega_g \mathcal{A}_a) g_{kc} - (\partial_k + \omega_g \mathcal{A}_k) g_{ca}] \zeta^c. \tag{7}$$

For tensors with rank bigger the one we have one combination of the connections for each Christoffel connection such that we can always combine one metric derivative with one Weyl connection as expressed above for the contra-variant vector. It means that in order to implement Weyl covariance we can just replace the partial derivative ∂_a by the Weyl invariant one:

$$\partial_a \mathcal{O} \to (\partial_a + \omega_{\mathcal{O}} \mathcal{A}_a) \mathcal{O}. \tag{8}$$

In this approach the invariance under local scaling is replaced by the manifestation of a one-form gauge field $\mathcal{A} = \mathcal{A}_a dx^a$. This gauge field does not represent physical degrees of freedom since it is a connection, the corresponding observables are the components of the gauge invariant tensor

$$\mathcal{F}_{ab} = \nabla_a \mathcal{A}_b - \nabla_b \mathcal{A}_a = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a. \tag{9}$$

2.1. Curvature

To show how this replacement works the commutation of the Weyl-covariant derivatives is calculated using this approach and will be shown that it provides correctly the well known result

$$[\mathcal{D}_a, \mathcal{D}_b] \zeta^c = \omega_\zeta \mathcal{F}_{ab} \zeta^c + \mathcal{R}_{ab}{}^c{}_d \zeta^d, \tag{10}$$

where $\mathcal{R}_{ab}{}^c{}_d$ is the conformal Riemann tensor defined by

$$\begin{aligned} \mathcal{R}_{ab}{}^c{}_d = {} & R_{ab}{}^c{}_d + \nabla_a [\delta_b{}^c \mathcal{A}_d + \delta_d{}^c \mathcal{A}_b - g_{bd} \mathcal{A}^c] \\ & - \nabla_b [\delta_a{}^c \mathcal{A}_d + \delta_d{}^c \mathcal{A}_a - g_{ad} \mathcal{A}^c] \\ & - [\delta_a{}^f \mathcal{A}_d + \delta^f{}_d \mathcal{A}_a - g_{ad} \mathcal{A}^f][\delta_b{}^c \mathcal{A}_f + \delta^c{}_f \mathcal{A}_b - g_{bf} \mathcal{A}^c] \\ & + [\delta_b{}^f \mathcal{A}_d + \delta^f{}_d \mathcal{A}_b - g_{bd} \mathcal{A}^f][\delta_a{}^c \mathcal{A}_f + \delta^c{}_f \mathcal{A}_a - g_{af} \mathcal{A}^c]. \end{aligned} \tag{11}$$

First note that the above equation can be obtained using the above prescription of minimal coupling. It is straightforward to show that the conformal Riemann tensor $\mathcal{R}_{ab}{}^c{}_d$ is obtained by taking the usual Riemann tensor expressed in terms of the derivatives of the metric field ∂g and performing the replacement of ∂g by $(\partial + \omega_g \mathcal{A})g$.

It is useful to look at the usual Christoffel connection as a vector operator Γ_a. This operator can act in arbitrary tensor, its action is defined as follows: for a contra-variant vector $\Gamma_a \zeta^b = \Gamma^b_{af} \zeta^f$, for a covariant vector $\Gamma_a \alpha_b = -\Gamma^f_{ab} \alpha_f$, for a rank 2 tensor $\Gamma_a G^b{}_c = \Gamma^b_{ak} G^k{}_c - \Gamma^k_{ac} G^b{}_k$ and so on. With this definition the geometrical covariant derivative takes the nice form $\nabla_a = \partial_a + \Gamma_a$. The usual Riemann tensor is defined by the action of the commutator of the diffeomorphic covariant derivative on a vector field $[\nabla_a, \nabla_b] \varphi^c = R_{ab}{}^c{}_d \varphi^d$ and can be expressed as follows

$$\begin{aligned} [\nabla_a, \nabla_b] \zeta^c &= [\partial_a + \Gamma_a, \partial_b + \Gamma_b] \zeta^c \\ &= ([\partial_a, \Gamma_b] + [\Gamma_a, \partial_b] + [\Gamma_a, \Gamma_b]) \zeta^c = R_{ab}{}^c{}_d \zeta^d. \end{aligned}$$

Apply the minimal coupling prescription means that the replacement $\partial_a \zeta \to (\partial + \omega_\zeta \mathcal{A}) \zeta$ and $\Gamma[\partial g] \to \bar{\Gamma} = \Gamma[(\partial + \omega_g \mathcal{A}) g]$ is performed and one find that

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b] \zeta^c = {} & \omega_\zeta ([\partial_a, \mathcal{A}_b] + [\partial_b, \mathcal{A}_a]) \zeta^c + \big([\partial_a, \bar{\Gamma}_b] + [\bar{\Gamma}_a, \partial_b] \\ & + [\bar{\Gamma}_a, \bar{\Gamma}_b]\big) \zeta^c. \end{aligned} \tag{12}$$

In first bracket appear the curvature for the gauge connection $\mathcal{F}_{ab} = [\partial_a, \mathcal{A}_b] + [\partial_b, \mathcal{A}_a]$. The second term on the hight hand side is the Riemann tensor calculated using the new Christoffel symbol $\bar{\Gamma}$, where the metric is minimally coupled to the Weyl