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A unifying framework for ghost-free Lorentz-invariant Lagrangian field theories

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ABSTRACT

We propose a framework for Lorentz-invariant Lagrangian field theories where Ostrogradsky's scalar ghosts could be absent. A key ingredient is the generalized Kronecker delta. The general Lagrangians are reformulated in the language of differential forms. The absence of higher order equations of motion for the scalar modes stems from the basic fact that every exact form is closed. The well-established Lagrangian theories for spin-0, spin-1, p-form, spin-2 fields have natural formulations in this framework. We also propose novel building blocks for Lagrangian field theories. Some of them are novel nonlinear derivative terms for spin-2 fields. It is nontrivial that Ostrogradsky's scalar ghosts are absent in these fully nonlinear theories.

1. Introduction

The problem of ghost-like degrees of freedom are encountered in the construction of theories with large numbers of spacetime indices, either from high spin fields or high order derivatives. High spin fields are dangerous because some tensor indices become derivative indices for the longitudinal modes, while high derivative Lagrangians are dangerous due to Ostrogradsky's instability.

A host of ghost-free theories were carefully constructed, resulting in the emergence of a common pattern. The linear theory of ghost-free massive gravity requires the Fierz–Pauli tuning [1], where the indices of mass terms are contracted antisymmetrically. When Lovelock studied the most general metric theories with second order equations of motion [2], the ghost-free combinations turned out to be antisymmetric products of Riemann curvature tensors. The antisymmetric structure appeared, again and again, in the high derivative scalar theories free of Ostrogradsky's ghost [3], nonlinear potential terms for massive spin-two fields [4], etc. [5–7].

This general pattern also applies to conventional theories. Both the Maxwell action for massless spin-1 fields and the linearized Einstein–Hilbert action for massless spin-2 fields are two-derivative quadratic actions with indices contracted antisymmetrically. Antisymmetrization seems to be a universal element in ghost-free Lorentz-invariant Lagrangians.

It is tempting to develop a unifying framework for local, ghost-free, Lorentz-invariant, Lagrangian field theories, with antisymmetrization being a key ingredient. It is clearly not an easy task to keep track of all the degrees of freedom in full generality and to make sure they are all free of ghost-like instability. As a first step, we will focus on the scalar modes. The absence of scalar ghosts is a necessity for completely ghost-free models. Therefore, ghost-free theories belong to a subset of scalar-ghost-free models and can be covered in this work.

In this work, we propose that the Lagrangians should be some differential forms. This may seem like a trivial statement as the volume element is itself a differential form and a Lagrangian, as the integrand of an action integral over spacetime, is defined as the product of a scalar function and the volume form.

The refinement in our proposal is that the Lagrangians should be the wedge products of geometric forms and matter forms

$$\mathcal{L} = \sum f\, \omega_1 \wedge \cdots \wedge \omega_n, \tag{1}$$

where the precise meanings of these differential forms ω_k are discussed later. In the simplest case, $\omega_k = E$ are the same vielbein one-form. Their wedge product gives the volume form, which appears in general relativity as the cosmological constant term. In general situations, ω_k could be the curvature two-forms and some exact forms constructed from matter fields. The use of these elabo-

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rate building blocks can help us to understand why higher derivative terms are forbidden from appearing in equations of motion: the absence of ghost-like degrees of freedom becomes a consequence of a basic property of exterior derivative

$$d^2 = 0, \tag{2}$$

whose geometric interpretation by Stokes's theorem is the boundary of a boundary vanishes.

2. Ostrogradsky's ghosts

In this section, let us explain the ghost problem in our discussions. Ostrogradsky's theorem states that the energy of a Lagrangian theory with higher order time derivative terms is not bounded from below because the Hamiltonian will be linear in a conjugate momentum [8]. The additional negative energy modes are ghost-like degrees of freedom and their presence is due to the fact that equations of motion are of higher than second order. A loophole in Ostrogradsky's proof is the assumption that the Lagrangian is non-degenerate. In other words, if the Euler–Lagrange equations remain second order, a higher derivative Lagrangian theory could be healthy and no additional problematic degree of freedom is propagating.

The absence of higher order time derivative terms in the equations of motion is not a sufficient condition for healthy models. For example, a Hamiltonian could be unbounded due to a wrong-sign kinetic term or potential term.

A more subtle point is that, in high spin field theories, the Hamiltonians can be unbounded from below, even if Ostrogradsky's ghosts are absent and the Lagrangian does not contain any wrong sign. An important example is the linearized Einstein–Hilbert term, whose Hamiltonian is unbounded due to the opposite signs of two momentum squared terms.[1,2]

Having these subtleties in mind, we would like to confine our attention to the absence of Ostrogradsky's ghosts in the sense that no additional degree of freedom is propagating and the equations of motion are at most of second order. Whether the Hamiltonians are bounded from below in the end is beyond this work. To further simplify our discussion, we will mainly concentrate on the scalar modes and require the absence of Ostrogradsky's scalar ghosts.

3. Lagrangians free of Ostrogradsky's scalar ghosts

Now we want to discuss general Lagrangians that could be free of Ostrogradsky's scalar ghosts. We will explain how the generalized Kronecker delta arises as a result of second order equations of motion together with Lorentz invariance. Then we will derive the general form of ghost-free Lorentz-invariant Lagrangians in the language of tensors.

By Lorentz-invariance, we mean that Minkowski vacuum is a solution of the models and, after we expand the Lagrangians around this background solution, the field contents transform properly under Lorentz transformations. The action should be invariant under these global symmetry transformations. So the theories do not distinguish among time and different space indices up some signs.[3] This definition of Lorentz-invariance applies to gravitational theories when the Minkowski metric is a solution.

A Lagrangian constructed from zeroth and first order terms will not lead to apparent higher order equations of motion. Let us consider a Lagrangian with harmless zeroth order terms and dangerous second order derivative terms

$$\mathcal{L} \sim \phi \dots \phi \, \partial\partial\phi \dots \partial\partial\phi, \tag{3}$$

where ϕ indicates dynamical fields and they can have tensor indices. Without $\phi \dots \phi$, the equations of motion will be of higher than second order or simply vanish. Higher than second order derivative terms are not considered because they usually lead to terms of at least the same order in the equations of motion after varying the $\phi \dots \phi$ part with respect to ϕ. We postpone the inclusion of first order derivative terms to later discussions.

Now we examine the variations of the product of two second order terms

$$\delta(\partial_{a_1}\partial_{a_2}\phi \, \partial_{b_1}\partial_{b_2}\phi \dots) \to \left(\partial_{a_1}\partial_{b_1}\partial_{b_2}\phi \, \partial_{a_2} \dots + \dots\right), \tag{4}$$

where we concentrate on a third order term on the right hand side. There are fourth order derivative terms as well, but the spirit manifests itself already in the third order terms. Since derivatives commute with each others, third order derivative terms with the same indices but different orders are equivalent. The coefficient of a third order derivative term is

$$C^{\mu,\nu\rho}\partial_\mu\partial_\nu\partial_\rho\phi = \left(C^{a,bc} + C^{a,cb} + C^{b,ac} + C^{b,ca} + C^{c,ab} + C^{c,ba}\right)\partial_a\partial_b\partial_c\phi. \tag{5}$$

To have a vanishing coefficient,[4] there are two simple choices: either (μ, ν) or (μ, ρ) are antisymmetrized. A more detailed derivation is given in section II of [9]. For the second cubic derivative term, we impose the same requirement, then we have two ansatzes to obtain second order equations of motion:

- (a_1, b_1) and (a_2, b_2) are two sets of antisymmetrized indices;
- (a_1, b_2) and (a_2, b_1) are two sets of antisymmetrized indices.

The same requirements for other second order derivative terms lead to two chains of antisymmetrized indices for the derivative indices of the second order terms. These tensor structures correspond to Young diagrams with two columns.

Let us now consider first order derivative term $\partial\phi$. At first sight, varying the first order term will lead to a third order term

$$\delta(\partial\phi)\,\partial\partial\phi \to -\partial\partial\partial\phi, \tag{6}$$

but, in the case of single scalar field, this term is canceled by varying the corresponding second order term

$$\partial\phi\,\delta(\partial\partial\phi) \to \partial\partial\partial\phi, \tag{7}$$

[1] The linear theories of massless higher spin fields may share the same issue. An explicit example can be found in [10], where the Fronsdal action [11] for a spin-3 field is rewritten in Hamiltonian form.

[2] In the Hamiltonian form of the Einstein–Hilbert action, the momentum squared terms are included in the Hamiltonian constraint and the Hamiltonian simply vanishes on the constraint surface.

[3] Some of our results can be generalized to other maximally symmetric vacua, i.e. de-Sitter space and Anti de-Sitter space. For example, the cosmological constant term mentioned in the introduction can lead to dS or AdS background solutions. Background-independent ghost-free theories should reduce to Lorentz-invariant ghost-free theories if the limit is not singular, so they could be obtained by proper generalizations of a subset of Lorentz-invariant ghost-free theories.

[4] We assume all higher order derivative terms should be absent, which seems to be stronger than the absence of higher order time-derivative terms. For example, $\partial_i\partial_0\partial_0\phi$ is of second order in time derivatives, but it comes from a third order term $\partial_\mu\partial_\nu\partial_\rho\phi$. If $\partial_0^3\phi$ is eliminated by some special tensor structure, then the third order term $\partial_\mu\partial_\nu\partial_\rho\phi$ is allowed.