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Differential expansion for link polynomials

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ABSTRACT

The differential expansion is one of the key structures reflecting group theory properties of colored knot polynomials, which also becomes an important tool for evaluation of non-trivial Racah matrices. This makes highly desirable its extension from knots to links, which, however, requires knowledge of the $6j$-symbols, at least, for the simplest triples of non-coincident representations. Based on the recent achievements in this direction, we conjecture a shape of the differential expansion for symmetrically-colored links and provide a set of examples. Within this study, we use a special framing that is an unusual extension of the topological framing from knots to links. In the particular cases of Whitehead and Borromean rings links, the differential expansions are different from the previously discovered.

1. Introduction

Knot theory is currently one of the main applications of quantum field theory, where non-perturbative results can be reliably derived and tested. One of the subjects to study is the representation dependence of Wilson loop averages (which, in four dimensions, distinguishes between the area and perimeter laws). The $3d$ Chern–Simons theory [1] underlying knot theory is topological, therefore, its observables can not depend on the metric data like lengths and areas, still their dependence on representations is quite non-trivial. Differential expansion [2–8] is the simplest manifestation of such properties. The goal of this paper is to extend the knowledge about differential expansion from knots to links. This is an essentially new story, because links consist of different components, each in its own representation. Calculation requires $\mathcal{R}$- and Racah matrices in channels with different representations, which are not yet well studied and where some useful methods like the eigenvalue conjecture [9] are not directly applicable. Important for differential expansion is the choice of framing. For knots, the best choice is the topological framing, when the $\mathcal{R}$-matrix is normalized in such a way that it provides invariance with respect to the first Reidemeister move. When the $\mathcal{R}$-matrix acts on a pair of different representations $R_1 \otimes R_2$, the first Reidemeister move is not applicable, and the topological framing is not defined from the first principles. Even in the case of links, there is a distinguished *canonical framing* (or standard framing) [10] suggested by M. Atiyah in [11]. It turns out, however, that the differential expansion requires a bit different framing.

We briefly remind what the differential expansion is in sec. 3 and formulate it for links of different kinds: made from unknots and from non-trivial knots. These conjectures are extracted from calculations of numerous examples, which became possible due to advances in Racah calculus in [12,13].

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2. Implications of representation theory

The HOMFLY-PT polynomials [14]

$$H_R^{\mathcal{K}}(A,q)\Big|_{q=e^{\frac{2\pi i}{k+N}},\ A=q^N} = \left\langle \mathrm{Tr}_R\ \mathrm{Pexp}\left(\oint_{\mathcal{K}} \mathcal{A}\right)\right\rangle^{CS_k^{SU(N)}} \tag{1}$$

are characters of the loop algebra in Chern–Simons theory [1] and inherit a lot of properties of characters of the ordinary groups. In particular, the antisymmetric representation $[1^r]$ and its conjugate $[1^{N-r}]$ are isomorphic in $SU(N)$, thus, the corresponding normalized HOMFLY-PT polynomials are the same in the topological framing:

$$\frac{H_{[1^r]}}{D_{[1^r]}} - \frac{H_{[1^{N-r}]}}{D_{[1^{N-r}]}} \ \vdots\ \{A/q^N\} \tag{2}$$

where $\{x\} \equiv x - \frac{1}{x}$ and $D_R(A,q) = \chi_R\left(p_k = \frac{\{Aq^k\}}{\{q^k\}}\right)$ are the values of Schur functions (characters) at the topological locus [15,16], for $A = q^N$ they turn into dimensions of representations of $SU(N)$, which are themselves invariant under conjugation. Strictly speaking, the difference at the l.h.s. needs just to vanish at $A = q^N$, the stronger statement with the factor $\{A/q^N\} \sim (A - q^N)(A^{-1} - q^{-N})$ is true only in the special (topological) framing. Since the transposition of representations is equivalent to the change $q \longrightarrow -q^{-1}$ in knot polynomials, as a corollary, we get for the symmetric representations $R = [r]$ and $[N-r]$ (which are no longer conjugate) and for arbitrary N:

$$\frac{H_r}{D_r} - \frac{H_{N-r}}{D_{N-r}} \ \vdots\ \{Aq^N\}\{A/q\} \tag{3}$$

The second factor at the r.h.s. reflects the triviality of reduced knot polynomials in the Abelian case $N = 1$, i.e. at $A = q$. This again depends on the choice of the topological framing, otherwise the items at the l.h.s. would be multiplied by different powers of A and q proportional to the writhe number, and (3) would be violated. The whole $\{A/q\}$ instead of just $(A - q)$ appears because the powers in A and q in all the terms of the knot polynomials have the same parity.

Our **first claim** in this paper is that there exists a framing in which a similar relation is true for links:

$$\frac{H_{r_1\ldots r_l}}{D_{r_1}\ldots D_{r_l}} - \frac{H_{N-r_1\ldots N-r_l}}{D_{N-r_1}\ldots D_{N-r_l}} \ \vdots\ \{Aq^N\}\{A/q\} \tag{4}$$

where l is the number of link components.

Our **second claim** is that these simple relations have far-going consequences, leading to a very powerful and restrictive representation of knot/link polynomials: the differential expansion. It works in an especially impressive way for knots in symmetric (and antisymmetric) representations, and we begin with reminding this part of the story. For an even more impressive (though far more involved) lift to rectangular representations, see [17]. The main part of the present letter is another extension: still symmetric representations, but for links. In principle, in this case, it is very useful to distinguish the link made from unknot components from those, where components are knotted themselves, though this issue is not that important as compared with the very fact of existence of a special framing which provides a **differential expansion for links**.

3. Differential expansion for knots in symmetric representations

As a corollary of (3), we obtain the following formula [3–8], which we call *differential expansion*:

$$\boxed{\mathcal{H}_r^{\mathcal{K}} = \frac{H_r^{\mathcal{K}}}{D_r} = 1 + \sum_{s=1}^{r} \frac{[r]!}{[s]![r-s]!} \cdot F_s^{\mathcal{K}}(A,q)\cdot\{A/q\}\cdot\prod_{j=0}^{s-1}\{Aq^{r+j}\}} \tag{5}$$

Indeed, $\mathcal{H}_0 = 1$, and (3) at $N = r$ implies that

$$\mathcal{H}_r = 1 + \{Aq^r\}\{A/q\}\cdot F_r \tag{6}$$

with some $F_r(q,A)$. At $N = r+1$, one gets additionally

$$\frac{\mathcal{H}_r - \mathcal{H}_1}{\{A/q\}} = \{Aq^r\}F_r - \{Aq\}F_1 \ \vdots\ \{Aq^{r+1}\} \implies F_r = [r]\cdot F_1 + \{Aq^{r+1}\}\tilde{F}_r \tag{7}$$

After this substitution at $N = r+2$,

$$\frac{\mathcal{H}_r - \mathcal{H}_2}{\{A/q\}} = \Big(\underbrace{[r]\{Aq^r\} - [2]\{Aq^2\}}_{[r-2]\{Aq^{r+2}\}}\Big)F_1 + \{Aq^r\}\{Aq^{r+1}\}\tilde{F}_r - \{Aq^3\}\{Aq^2\}\tilde{F}_2 \ \vdots\ \{Aq^{r+2}\} \implies$$

$$\implies \tilde{F}_r = \frac{[r][r-1]}{[2]}\tilde{F}_2 + \{Aq^{r+2}\}\tilde{\tilde{F}}_r \tag{8}$$

where the coefficient $\frac{[r][r-1]}{[2]}$ is the value of $\{Aq^3\}\{Aq^2\}$ at $A = q^{-r-2}$. Repeating the same procedure for higher and higher N and adjusting the notation, one gets (5).