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Nonexistence of degenerate horizons in static vacua and black hole uniqueness

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ABSTRACT

We show that in any spacetime dimension $D \geq 4$, degenerate components of the event horizon do not exist in static vacuum configurations with positive cosmological constant. We also show that without a cosmological constant asymptotically flat solutions cannot possess a degenerate horizon component. Several independent proofs are presented. One proof follows easily from differential geometry in the near-horizon limit, while others use Bakry–Émery–Ricci bounds for static Einstein manifolds.

1. Introduction and main results

In the classical proof of static vacuum black hole uniqueness, the last case to be considered was that in which degenerate components of the event horizon were present. As was shown in [3,4], such black hole configurations cannot occur. This result applies to the 4-dimensional setting with vanishing cosmological constant $\Lambda = 0$. The authors in [4] also obtained certain restrictions in the higher dimensional setting and in the presence of a nonzero cosmological constant, but were ultimately unable to extend their result to these situations. The purpose of the present paper is to do just that for $\Lambda \geq 0$. The main result is as follows.

Theorem 1.

(i) *There do not exist static vacuum black holes having a degenerate horizon component in the presence of a positive cosmological constant $\Lambda > 0$.*
(ii) *A complete solution of the static vacuum equations with $\Lambda = 0$ can have no more than one connected component of a degenerate horizon.*
(iii) *A solution of the static vacuum equations with $\Lambda = 0$ and having an asymptotically flat end[1] cannot have a degenerate horizon component.*

An immediate consequence of part (iii) is a generalized version of the classical static black hole uniqueness result. In dimensions $D > 4$ the uniqueness proofs [6–8] rely on the positive mass theorem and require all horizon components to be nondegenerate. Here we have shown that degenerate components do not exist, which when combined with [6–8] leads to the following statement.

Theorem 2. *In any dimension $D \geq 4$, an asymptotically flat static vacuum black hole is isometric to a Schwarzschild–Tangherlini solution.*

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$ on which a function φ is defined. Consider the associated static spacetime $(\mathbb{R} \times M^n, G)$ where the spacetime metric takes the form

$$G = -e^{-2\varphi} dt^2 + g. \tag{1.1}$$

It is assumed that the lapse function is positive on M^n (hence we write it as e^{φ}), and vanishes on the topological boundary $\partial M^n = \overline{M}^n \setminus M^n$ which itself should be a compact smooth manifold. The vacuum Einstein equations

$$\mathrm{Ric}(G) - \frac{1}{2} R(G) G + \Lambda G = 0, \tag{1.2}$$

are equivalent to the following set of equations on M^n

$$\begin{aligned} &\mathrm{Ric}(g) + \mathrm{Hess}_g \varphi - d\varphi \otimes d\varphi = \frac{2\Lambda}{n-1} g, \\ &\Delta_g \varphi - |d\varphi|_g^2 = \frac{2\Lambda}{n-1}. \end{aligned} \tag{1.3}$$

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[1] This refers to the standard notion of asymptotic flatness within the black hole uniqueness context, see e.g. [7].

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The second equation carries little independent information, and is only slightly more than an integrability condition for the first. The first equation together with the twice contracted second Bianchi identity shows that

$$\Delta_g \varphi - |d\varphi|^2_g = \frac{2\Lambda}{n-1} + Ce^{2\varphi} \,, \tag{1.4}$$

where C is a constant. The second equation of (1.3) is recovered in the case that $C = 0$.

Recall that a Killing horizon is a null hypersurface defined by the vanishing in norm of a Killing field V, which is normal to the horizon. In the static case above $V = \partial_t$ and the Killing horizon corresponds to $\mathbb{R} \times \partial M^n$. Killing horizons come naturally equipped with a notion of surface gravity κ, defined through the equation

$$d|V|^2 = -2\kappa V \,. \tag{1.5}$$

A component of the horizon is referred to as degenerate (or extreme) if its surface gravity vanishes $\kappa = 0$.

An important observation is that the static vacuum equations can be expressed in terms of the N-Bakry–Émery–Ricci tensor

$$\mathrm{Ric}^N_\varphi(g) := \mathrm{Ric}(g) + \mathrm{Hess}_g \varphi - \frac{1}{(N-n)} d\varphi \otimes d\varphi \,. \tag{1.6}$$

In general N may take values in the compactified real line, where the last term in (1.6) is removed when $N = \pm\infty$. When N is an integer and greater than n, this expression arises when an N-dimensional metric splits as a warped product over (M^n, g). The N-dimensional Ricci tensor splits and its projection onto the base yields Ric^N_φ. Specifically, if $M^n \times \mathcal{F}^{N-n}$ has metric

$$G = g_M \oplus e^{-2\varphi/(N-n)} g_{\mathcal{F}} \,, \qquad \varphi : M \to \mathbb{R} \,, \tag{1.7}$$

then (since g_M is independent of $\mathcal{F}$ and $g_{\mathcal{F}}$ is independent of M)

$$\begin{aligned} \mathrm{Ric}(G) &= \left[\mathrm{Ric}(g_M) + \mathrm{Hess}_{g_M} \varphi - \frac{1}{(N-n)} d\varphi \otimes d\varphi \right] \oplus \left[\mathrm{Ric}(g_{\mathcal{F}}) \right. \\ &\quad \left. + \frac{1}{(N-n)} e^{-2\varphi/(N-n)} g_{\mathcal{F}} \left(\Delta_{g_M} \varphi - |d\varphi|^2_{g_M} \right) \right] \\ &= \mathrm{Ric}^N_\varphi(g_M) \oplus \left[\mathrm{Ric}(g_{\mathcal{F}}) \right. \\ &\quad \left. + \frac{1}{(N-n)} e^{-2\varphi/(N-n)} g_{\mathcal{F}} \left(\Delta_{g_M} \varphi - |d\varphi|^2_{g_M} \right) \right] . \end{aligned} \tag{1.8}$$

The term *synthetic dimension* for N arises since, in this context, N is the dimension of the total space. Note that the leaves M^n lie in $M^n \times \mathcal{F}^{N-n}$ as totally geodesic submanifolds (i.e., having vanishing second fundamental form). If $\mathcal{F}$ is 1-dimensional then $\mathrm{Ric}(g_{\mathcal{F}}) = 0$, $N = n+1$, and equation (1.2), written in the form $\mathrm{Ric}(G) = \frac{2}{(N-2)} \Lambda G$, splits into equations (1.3). Thus the first static vacuum equation in (1.3) may be rewritten as

$$\mathrm{Ric}^{n+1}_\varphi(g) = \frac{2\Lambda}{(n-1)} g. \tag{1.9}$$

Metrics which satisfy this relation are referred to as *quasi-Einstein* [2]. It turns out that many of the basic Ricci curvature results of Riemannian geometry are known to hold as well for the Bakry–Émery–Ricci curvature. In particular we will exploit Bakry–Émery versions of Myers' Theorem, the Splitting Theorem, and arguments used in the proof of Synge's Theorem [12]. It is the purpose of this paper to introduce these techniques into the study of static black hole uniqueness, and thereby establish Theorem 1. More precisely, Myers' Theorem and the Splitting Theorem will yield special cases of Theorem 1 in Section 3, and in Section 4 the Synge type methods will produce a full proof. Section 2 is dedicated to recording technical results concerning degenerate horizons for use in later sections. We note that the theory associated with Bakry–Émery–Ricci curvature has previously been applied to study static solutions of the Einstein equations which are geodesically complete, in [1,14,15].

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2. Degenerate components of the horizon

2.1. Degenerate horizons as asymptotic ends

Consider a degenerate component of the Killing horizon in a static black hole spacetime. A key prerequisite for application of the Riemannian geometric techniques mentioned in the introduction, is the fact that within time-symmetric slices such degenerate components lie infinitely far away from any other point. This has been shown in [3], although here we offer a simple proof using Gaussian null coordinates [10]. These coordinates may be introduced near a degenerate horizon, and give the following form of the spacetime metric

$$G = 2dv \left(dr + \frac{1}{2} r^2 F(r,x) dv + r h_a(r,x) dx^a \right) + h_{ab}(r,x) dx^a dx^b. \tag{2.1}$$

In keeping with conventions commonly used in near horizon geometry, here and onward $V = \partial_v$ represents the timelike Killing field (∂_t in the coordinates of equation (1.1)), $r = 0$ coincides with the horizon, and h_{ab} denotes the induced metric on $\mathcal{H}$ the horizon cross-section. The orbit space metric on a constant time slice M^n is then given by

$$g_{ij} = G_{ij} - \frac{G_{iv} G_{jv}}{G_{vv}} = G_{ij} + \frac{G_{iv} G_{jv}}{r^2 |F|}, \tag{2.2}$$

with coordinates $\mathbf{x}^i = (r, x^a)$. Note that since the Killing field is timelike away from the horizon

$$F(r,x) < 0 \qquad \text{for} \qquad r > 0. \tag{2.3}$$

It follows that

$$\begin{aligned} g(\partial_r, \partial_r) &= \frac{G_{rv}^2}{r^2|F|} = \frac{1}{r^2|F|}, \\ g(\partial_r, \partial_{x^a}) &= \frac{G_{rv} G_{av}}{r^2|F|} = -\frac{r h_a}{r^2|F|}, \\ g(\partial_{x^a}, \partial_{x^b}) &= G_{ab} + \frac{G_{av} G_{bv}}{r^2|F|} = h_{ab} + \frac{h_a h_b}{|F|}. \end{aligned} \tag{2.4}$$

Let $\gamma(r) = (r, x^a(r))$, $r \in [0, r_0]$ be a smooth curve in the orbit space intersecting $\mathcal{H}$, with tangent vector $\dot{\gamma}$. On this curve $|h_a \dot{x}^a| + |F| \leq$