Physics Letters B 777 (2018) 260–264

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Renormalization group procedure for potential $-g/r^2$

S.M. Dawid [a,b,*], R. Gonsior [a], J. Kwapisz [a], K. Serafin [a], M. Tobolski [c,a], S.D. Głazek [a,d]

[a] Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland
[b] Department of Physics, Indiana University, Bloomington, IN 47405, USA
[c] Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland
[d] Department of Physics, Yale University, New Haven, CT 06520, USA

ARTICLE INFO

Article history:
Received 27 April 2017
Received in revised form 26 August 2017
Accepted 12 December 2017
Available online 14 December 2017
Editor: J.-P. Blaizot

Keywords:
Renormalization group procedure
Hamiltonian
Limit cycle
Asymptotic freedom
Triviality
Fixed point
Scale symmetry breaking
Quantum mechanics

ABSTRACT

Schrödinger equation with potential $-g/r^2$ exhibits a limit cycle, described in the literature in a broad range of contexts using various regularizations of the singularity at $r = 0$. Instead, we use the renormalization group transformation based on Gaussian elimination, from the Hamiltonian eigenvalue problem, of high momentum modes above a finite, floating cutoff scale. The procedure identifies a richer structure than the one we found in the literature. Namely, it directly yields an equation that determines the renormalized Hamiltonians as functions of the floating cutoff: solutions to this equation exhibit, in addition to the limit-cycle, also the asymptotic-freedom, triviality, and fixed-point behaviors, the latter in vicinity of infinitely many separate pairs of fixed points in different partial waves for different values of g.

1. Introduction

Eigenvalue problem for the one-particle Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{g}{\vec{r}^{\,2}} \,, \tag{1}$$

provides a well-known example of a singular Schrödinger equation. For positive and sufficiently large coupling constant g, the overwhelmingly attractive potential causes instability, which limits application of Eq. (1) in physics [1].

The situation is changed when one regularizes the potential and introduces corrections on the basis of demanding that predictions for observables do not depend on the regularization. As a result, the corrected interaction exhibits cyclic behavior as a function of the regularization cutoff parameter. The cycle is associated with an infinite set of bound states whose binding energies form a geometric sequence converging on zero [1–9].

Regularization of potential $-g/r^2$ in the position representation is proposed in various ways [2,3,5,10]. For instance, Braaten and Phillips [5] cut off the potential at some small radius and they introduce an additional potential that acts only in a spherical δ-shell of an infinitesimally smaller radius. They solve the resulting s-wave eigenvalue problem and find that one can obtain cutoff independent eigenvalues by making the coupling constant in the additional term a log-periodic function of the cutoff radius.

In the momentum representation, the regularization is formulated differently. For example, in the space of s-wave states, Hammer and Swingle [6] introduce an ultraviolet cutoff Λ that limits the particle momentum from above. They also add a counter potential with a new coupling constant, $H(\Lambda)$. They determine $H(\Lambda)$ as a function of Λ by demanding that the bound-state solution with zero binding energy does not depend on Λ. Knowing the function $H(\Lambda)$, they find a differential equation that it satisfies. In higher partial waves, Long and van Kolck [8] discuss the potential $-g/r^2$ with added contact terms, in the spirit of constructing effective theories. They arrive at cutoff-independent solutions for observables by specifying cycling coefficients in front of one contact term per singular partial wave.

The approach quoted above has successfully amended Eq. (1) with examples of regularization and corresponding correction terms that are justified *a posteriori*. The ultimate justification relies on the fact that different regularizations with different corrections lead to the same results for observables [11,12].

In this paper, the Hamiltonian of Eq. (1) is handled using the Wilsonian type of renormalization group procedure [13,14]. It is

* Corresponding author.
E-mail address: sdawid@iu.edu (S.M. Dawid).

https://doi.org/10.1016/j.physletb.2017.12.028

the same type of procedure that originally produced the concept of a renormalization group limit cycle [15]. Instead of first calculating observables in a regularized theory, guessing an ansatz for the correction term, and subsequently checking if such a theory can give regularization independent results for the observables, we begin by *calculating* the required counterterm structure in the presence of regularization. This is done using a renormalization group transformation (RGT) of the Hamiltonian itself [13,14], *i.e.*, prior to seeking solutions for observables. In the process, a family of finite, effective Hamiltonians is obtained, using the calculated counterterm structure. Solutions for observables are sought first in the resulting finite, scale-dependent effective theories that *a priori* do not depend on regularization.

Our procedure leads to the limit cycle in terms of solutions to a simple renormalization group equation that evolves a Hamiltonian in the sense of Refs. [13,14]. The procedure also allows us to identify a whole range of renormalization group behaviors, which characterize the Hamiltonian of Eq. (1), in addition to the limit cycle. These other behaviors, to the best of our knowledge, are not fully identified in the literature. Namely, we exhibit behaviors of the asymptotic-freedom, triviality and fixed-point type. The latter has been found before in the case of s-waves, using the functional renormalization group technique that revealed a collision of two fixed points at the critical value of the s-wave coupling constant [16]. We identify fixed points in all partial waves, including their behavior near the corresponding infinitely many critical values of the coupling constant. These behaviors appear in a pattern of interest for studies of scaling symmetry and its breakdown in complex theories, see below. Hence, the simplicity and familiarity of Eq. (1) are its assets rather than drawbacks. Further simplification is taken advantage of in this paper by carrying out the RGT in the differential steps that are analogous to the discrete ones outlined in [17,18] on another example of a Hamiltonian with a limit cycle. The issue of dependence of effective theories on the magnitude of eigenvalues they aim to describe, is only briefly explained at the end of the paper.

Effective potentials of type $1/r^2$ were successfully employed in three-body dynamics, where a series of bound states generically appears, under the name of Efimov effect [19–22], whenever in the corresponding two-body dynamics the ratio of effective interaction range to scattering length is very small, ultimately requiring precise treatment when this ratio approaches zero [23]. It is worth pointing out that a potential of the type $1/r^2$ appears also in interactions of a point charge with a dipole, causing violation of scaling symmetry [24,25]. The pattern of breaking scale invariance when the coupling constant g in Eq. (1) increases above a critical value is of special interest in many areas "from molecular to black-hole physics" [26], and in statistical mechanics [27,28]. In Ref. [10], the analogous pattern is found helpful in discussing theories that exhibit the Berezinsky–Kosterlitz–Thouless phase transition. It is said to have many "parallels in the AdS/CFT correspondence" [29]. Patterns of conformal symmetry breaking, including breaking of scale invariance in Hamiltonians [30], are invoked in light-front holographic description of hadrons [31] and in studies of conformal windows in technicolor gauge theories [32], the latter hoped to help in understanding the origin of the Standard Model [33,34]. Equation (1) is thus of broad interest as a rich but simple example of scaling-symmetry breaking and dimensional transmutation [35] in the Wilsonian renormalization group procedure for Hamiltonians.

2. Renormalization group transformation

The renormalization group procedure for an incompletely defined Hamiltonian, such as the one in Eq. (1), starts with regularizing it as an operator in a scheme called the "triangle of renormalization" [36]. We introduce a cutoff Δ and thus define H_Δ, from which we evaluate the equivalent effective Hamiltonians H_Λ with cutoffs $\Lambda \ll \Delta$. The RGT is involved in the process of this evaluation.

In the limit $\Delta \to \infty$, we demand that matrix elements of H_Λ in the subspace of states limited by finite Λ do not depend on Δ. This condition implies a large, if not infinite set of constraints. It allows us to determine the structure of counterterms needed in H_Δ, up to finite alterations in their parameters. The computation of counterterms is done in a sequence of successive approximations, which improves the counterterms put in H_Δ until all matrix elements in H_Λ in the subspace of states limited by Λ become independent of Δ. This finite cutoff becomes a running parameter which enables one to identify how the theoretical description of observable phenomena depends on the range of scales of degrees of freedom one uses for constructing solutions of the theory.

Once the counterterms in H_Δ are such that all matrix elements of H_Λ have well-defined limits for $\Delta/\Lambda \to \infty$, the eigenvalues of H_Λ cannot depend on Δ. At the same time, eigenvalues of the effective Hamiltonians H_Λ that are much smaller than Λ cannot depend on Λ, because this cutoff is merely a mathematical boundary between the implicit degrees of freedom above and explicit degrees of freedom below Λ. Computation of H_Λ that is equivalent to H_Δ is carried out in a sequence of discrete, or infinitesimal RGTs.

We carry out a sequence of the RGTs in momentum representation. The stationary Schrödinger equation for Hamiltonian of Eq. (1),

$$\frac{\vec{p}^2}{2m}\phi(\vec{p}) - \frac{g}{4\pi}\int d^3q\, \frac{\phi(\vec{q})}{|\vec{p}-\vec{q}|} = E\phi(\vec{p}) \tag{2}$$

is written in terms of angular and radial variables, $\phi(\vec{p}) = \sum_{lm}\psi_l(p)Y_{lm}(\Omega_p)$, where $p = |\vec{p}\,|$. Taking advantage of rotational symmetry of Eq. (2), one obtains

$$p^2\psi_l(p) + \int_0^\infty dq\, q^2\, V_l(p,q)\, \psi_l(q) = \mathcal{E}\psi_l(p), \tag{3}$$

where the eigenvalue $\mathcal{E} = 2mE$, the potential

$$V_l(p,q) = -\frac{\alpha}{2l+1}\left[\frac{\theta(p-q)\, q^l}{p^{l+1}} + \frac{\theta(q-p)\, p^l}{q^{l+1}}\right], \tag{4}$$

the dimensionless coupling constant $\alpha = 2mg$ and the symbol θ denotes the Heaviside step function. Our regulated eigenvalue problem is defined by limiting the range of momenta p and q by a large cutoff parameter Δ.

Elimination of high momentum modes proceeds by infinitesimal steps such as from Δ to $\Delta - d\Delta$,

$$p^2\psi_l(p) + \int_0^{\Delta-d\Delta} dq\, q^2\, V_l(p,q)\, \psi_l(q) - \frac{\alpha}{2l+1}\frac{p^l}{\Delta^{l-1}}\,\psi_l(\Delta)\, d\Delta = \mathcal{E}\,\psi_l(p)\,. \tag{5}$$

The value of $\psi_l(\Delta)$ is expressed in terms of values of $\psi_l(p)$ with $p < \Delta - d\Delta$ by setting $p = \Delta$ in Eq. (5). For eigenvalues $\mathcal{E}$ much smaller than Δ^2, and neglecting terms that lead to quantities of order $d\Delta^2$ or smaller in Eq. (5), we have

$$\psi_l(\Delta) = \frac{\alpha}{(2l+1)\Delta^2}\int_0^{\Delta-d\Delta} dq\, q^2\, \frac{q^l}{\Delta^{l+1}}\,\psi_l(q)\,. \tag{6}$$