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Through the big bang: Continuing Einstein's equations beyond a cosmological singularity

Tim A. Koslowski [a], Flavio Mercati [b,c,*], David Sloan [d]

[a] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Cd. Universitaria, 04510 Ciudad de México, D.F., Mexico
[b] Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada
[c] Dipartimento di Fisica, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy
[d] Beecroft Institute of Particle Astrophysics and Cosmology, Department of Physics, University of Oxford, Denys Wilkinson Building, 1 Keble Road, Oxford OX1 3RH, UK

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A B S T R A C T

All measurements are comparisons. The only physically accessible degrees of freedom (DOFs) are dimensionless ratios. The objective description of the universe as a whole thus predicts only how these ratios change collectively as one of them is changed. Here we develop a description for classical Bianchi IX cosmology implementing these relational principles. The objective evolution decouples from the volume and its expansion degree of freedom. We use the relational description to investigate both vacuum dominated and quiescent Bianchi IX cosmologies. In the vacuum dominated case the relational dynamical system predicts an infinite amount of change of the relational DOFs, in accordance with the well known chaotic behaviour of Bianchi IX. In the quiescent case the relational dynamical system evolves uniquely though the point where the decoupled scale DOFs predict the big bang/crunch. This is a non-trivial prediction of the relational description; the big bang/crunch is not the end of physics – it is instead a regular point of the relational evolution. Describing our solutions as spacetimes that satisfy Einstein's equations, we find that the relational dynamical system predicts two singular solutions of GR that are connected at the hypersurface of the singularity such that relational DOFs are continuous and the orientation of the spatial frame is inverted.

The generic nature of singularities beyond which physics cannot be deterministically continued is a cornerstone of General Relativity (GR). The Hawking–Penrose theorems [1] show that a large class of solutions of Einstein's equations are geodesically incomplete. In cosmological settings this leads to the big bang (or crunch) – the inevitable end of classical evolution of the Lorentzian spacetime geometry. To establish this result, it is important to disentangle singularities of spacetime geometry from singular evolution. The classical singularity theorems are derived from a contradiction that arises between the properties of maximal time-like geodesics in Lorentzian spacetimes and the properties of time-like (or null) geodesics that can be derived from Einstein's equations for generic initial conditions when matter satisfies suitable energy conditions after finite proper time (or affine parameter). This leads to the conclusion that Einstein's equations predict the breakdown of spacetime geometry. What is not implied by these theorems is that the evolution of the dynamical system that describes the physical observables has to break down. These physical observables are far fewer than the auxiliary structure that is needed to describe spacetime geometry: In particular the lapse function, shift vector and spatial conformal factor are identified as non-dynamical in York's (see e.g. [2]) canonical description of GR in which only the pure spin-2 part of the metric evolves with matter. We refer to a dynamical system that evolves the observables of GR without reference to auxiliary spacetime structure; a relational description of gravity. This raises the possibility that the celebrated singularity theorems of GR solely predict the breakdown of auxiliary, non-dynamical structure, while they do not predict the breakdown of the relational system, so evolution equations for physical observables remain predictive in the sense that initial data determines a unique solution. In this letter we explore precisely this possibility in the relational description of homogeneous cosmology, where the singularity theorems predict a big bang or big crunch singularity of spacetime geometry. In particular, we show in this letter that when one considers cosmology from a relational perspective –

* Corresponding author.
E-mail address: flavio.mercati@gmail.com (F. Mercati).

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constructing only observables available *within* a toy model universe then the resulting dynamical system evolves all observables. We show that there exists a unique, deterministic, and entirely classical extension of Einstein's equations through the big bang/crunch. We achieve this result without appealing to quantum effects or new ad-hoc principles. Rather, the strict insistence on describing the dynamics in terms of relational variables alone ensures the existence and uniqueness of the evolution through the apparent singularity. The relational system predicts that the other side of the apparent singularity is a qualitatively similar yet quantitatively distinct cosmology *inverted in spatial orientation.* Space–time encounters a curvature singularity but the relational system remains well defined throughout. We establish our result through a three-step process. First we rewrite the dynamics of a homogeneous (but not necessarily isotropic) cosmology entirely in terms of relational variables. Second, we observe that the relational degrees of freedom form an autonomous subsystem that decouples from the evolution of the total size of the universe. Third, we show that the relational dynamical system remains deterministic while the system encounters (in finite physical time) a point at which Einstein's equations become singular.

1. Relational description of physics

The description of the universe as a whole can not depend on external units of length or duration since all physical clocks and rods are part of the universe itself. The universe differs fundamentally from its subsystems in this aspect. The problem with GR's distance defining spacetime geometry is thrown into focus when considering cosmology. In a laboratory experiment an observer can easily justify the separation between the measuring apparatus (the external clock and rod) and the objects being measured. Cosmologists however are part of the universe and can not separate themselves form the studied system. Cosmological measurements are intrinsic, as rods and clocks are constructed from the dynamical objects in the universe. Units are constructed intrinsically using physical reference structures that define GR's notion of geometry. Hence, all dimensional quantities are intrinsic ratios. This leads us to concentrate on the dynamics of *relational variables, i.e.* dimensionless ratios and their relative infinitesimal variations[1] [5]. Remarkably, the dynamics can be expressed entirely in terms of relational variables, which turn out to evolve autonomously and predict all *intrinsic observables* of GR.

In this letter we study spatially homogeneous cosmology (Bianchi IX) with a massless scalar field. This model is believed to correctly describe the near-singularity behaviour of full GR due to the BKL conjecture and Wheeler's insight that "matter doesn't matter" except a stiff component (such as a massless scalar field). In fact, the theorems of [3], show that a dense set of inhomogeneous GR solutions obey the BKL conjecture, *i.e.* spatial points decouple in the approach to the singularity and evolve as independent Bianchi IX systems. Moreover, a massless scalar field is compatible with Standard Model physics[2] (*e.g.* the Goldstone mode of the Higgs field). We thus distinguish two cases:

1. In absence of a massless scalar the approach to the singularity is given by the vacuum Bianchi IX evolution, in which the dynamics never actually reaches the singularity. It rather goes through an infinite amount of change, with infinitely many billiard-like 'bounces' against steep triangular potential walls, alternating with intervals of free geodesic evolution (Kasner epochs). This fact was observed by Misner [7], and its consequences for the status of the singularity was discussed in [4]. This has an important consequence in the relational framework, where physical clocks necessarily possess internal relational DOFs which register time. An infinitesimal clock can not be treated as the idealized worldline of a point with its proper time, but has to be viewed as the infinitesimal limit of a sequence of ever smaller time-recording systems with internal structure [8]. It has been noted [8] that the change of the internal relational DOFs of an infinitesimal limit clock will be subject to the same tidal effects as measured by its large counterparts. It follows that infinitesimal clocks register an unbounded lapse of time (*i.e.* change of internal relational DOFs), when the gravitational field experiences an infinite number of Kasner epochs. It follows that the infinitesimal clocks, unlike their pointlike idealizations *i.e.* proper time, will not reach the big bang/crunch in a finite time.
2. In the presence of a massless scalar one experiences "quiescent" behaviour [5]. The potential becomes irrelevant for the dynamics and the equations of motion asymptote into a geodesic evolution. Matter clocks will measure a finite amount of change between the singularity and any other point. It remains to investigate this case, because it is the one in which the singularity is reached in finite relational time and we have to establish what happens to the relational DOFs there.

2. Quiescent Bianchi IX cosmology

Using triad variables, we describe Bianchi IX cosmology, *i.e.* homogeneous geometries on S^3, as:

$$ds^2 = -dt^2 + \delta_{ab} e^a_i\, e^b_j\, dx^i dx^j \,. \tag{1}$$

Imposing translational invariance, and fixing a global $SO(3)$ rotation, we write $e^a_i dx^i = \pm v^{1/3} e^{\gamma_a} \sigma_a$, where v is the spatial volume, γ_a are three anisotropy parameters constrained by $\gamma_1 + \gamma_2 + \gamma_3 = 0$, and σ_a are the three translation-invariant one-forms on S^3. The $\pm$ in the definition of e^a_i refers to the orientation of the spatial manifold and does not enter the metric. We can locally parametrize the anisotropy parameters with two Misner variables q^1, q^2, defined as: $\gamma_1 = -q^1/\sqrt{6} - q^2/\sqrt{2}$, $\gamma_2 = -q^2/\sqrt{6} + q^1/\sqrt{2}$, $\gamma_3 = \sqrt{\tfrac{2}{3}} q^2$, which coordinatize half of shape space. Useful global coordinates for shape space are the angles (α, β), defined by

$$\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = |\tan\beta| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad \operatorname{sign}(\det e) = \operatorname{sign}(\tan\beta)\,, \tag{2}$$

where the sign of β represents the two possible orientations of e^a_i. (α, β) are spherical coordinates for the representation of shape space shown in Fig. 1.

We denote the shape momenta canonically conjugate to q^a by p_a, the variable conjugate to v by τ, the (homogeneous) massless scalar field by φ and its momentum density by π. Due to dynamical similarity [9,10], only the latter will appear in the equations. Einstein's equations are generated by the ADM Hamiltonian [11] (which is constrained to vanish)

$$H = p_1^2 + p_2^2 + \frac{\pi^2}{2} - \frac{3}{8}\,\tau^2\, v^2 - v^{\frac{4}{3}}\, C(q^1, q^2) \approx 0, \tag{3}$$

where $C(q^1, q^2)$, the shape potential shown in Fig. 1, is

$$\begin{aligned} C(q^1, q^2) &= F(2q^2) + F(q^1\sqrt{3} - q^2) + F(-q^1\sqrt{3} - q^2)\,, \\ F(x) &= e^{-x/\sqrt{6}} - \tfrac{1}{2} e^{2x/\sqrt{6}}\,. \end{aligned} \tag{4}$$

The equations of motion [using a vector notation $\vec{q} = (q^1, q^2)$, $\vec{p} = (p_1, p_2)$] are

[1] *E.g.* the scale factor is *not* a relational variable, while Misner's anisotropy parameters β_+, β_- [4] and the ratio of their variations $d\beta_+/d\beta_-$ are relational variables.

[2] Interestingly, an RG improved gravitational action, as obtained in the functional renormalization group setting, also offers a mechanism to achieve this quiescent behavior [6].