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Non-extensive statistical mechanics and black hole entropy from quantum geometry

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ABSTRACT

Using non-extensive statistical mechanics, the Bekenstein–Hawking area law is obtained from microstates of black holes in loop quantum gravity, for arbitrary real positive values of the Barbero–Immirzi parameter (γ). The arbitrariness of γ is encoded in the strength of the "bias" created in the horizon microstates through the coupling with the quantum geometric fields exterior to the horizon. An experimental determination of γ will fix this coupling, leaving out the macroscopic area of the black hole to be the only free quantity of the theory.

1. Introduction

Loop quantum gravity (LQG) provides us with an estimate of the microstates of a black hole with given classical area (A), albeit at the kinematical level, which leads to a precise computation of its entropy [6,7]. The whole procedure is completed in two steps. Step 1: One deals with the field dynamics on the horizon to unravel the nature of the Hilbert space, hence the microstates, associated with the same. Step 2: Having the estimate of the microstates at hand, one applies the statistical mechanics to calculate the entropy (S).

The calculation yields $S = (\lambda_0/\gamma)A$, where γ is known as the Barbero–Immirzi parameter [9–11], λ_0 is a numerical constant resulting from the underlying statistical mechanics. Then, demanding that S be given by the Bekenstein–Hawking area law (BHAL) i.e. $A/4$ [4,5], the value of γ is suitably fixed. Now, the parameter γ represents a one-parameter family of canonical transformation of the canonical variables of the classical theory i.e. for every value of γ the classical equations of motion of general relativity are valid. And, in the quantum theory γ represents a quantization ambiguity of the theory like the θ-parameter of quantum chromodynamics. For every real and positive[1] value of γ there is a valid quantum theory, but they are unitarily inequivalent. In principle, if the derivation of the black hole entropy is correct, then the BHAL should follow for all real and positive values of γ. Hence, the necessity of choosing γ by hand just for the sake of obtaining the BHAL is considered as a drawback of this approach of black hole entropy calculation [12]. It is suggested that one should obtain the $S = A/4$ for arbitrary γ. Hence, a physical explanation behind the tuning of γ and obtaining the BHAL for arbitrary γ, both will be value additions to the concerned literature. We achieve these goals by introducing a statistical mechanics or rather definition of entropy other than the one, namely Boltzmann entropy, used in the standard literature to calculate black hole entropy from LQG. So, let us provide the physical motivation behind doing that on the first place.

2. Motivation

The physical motivation behind changing the standard definition of entropy comes from an observation regarding the two steps involved in the black hole entropy calculation in LQG.

Step 1 consists of the exploration of the quantum field dynamics on the horizon. Classically, it has been shown for the case of Schwarzschild black hole that the field equations on the horizon can be derived from the action of a CS theory coupled to an external source [16]

$$S_{CS} = \frac{k}{4\pi}\int \epsilon^{abc}\left(A_a^I\partial_b A_c^I + \frac{1}{3}\epsilon^{IJK}A_a^I A_b^J A_c^K\right) + \int J^{Ia}A_a^I \quad (1)$$

where A_a^I is the CS gauge field, J^{Ia} is the external source that is dual to the bulk soldering form both with respect to the internal indices and the spacetime indices; I, J, K represent the internal

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[1] γ should be real and positive in the quantum theory because it appears as a multiplicative constant in the area spectrum of the black hole [6].

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gauge index and a, b, c represent the spacetime indices. The coupling with the external source represents the coupling between the horizon and the external bulk. In the quantum theory these sources represent point-like quantum geometric excitations on the horizon [6]. The Hilbert space associated with the black hole horizon is that of a Chern–Simons (CS) theory coupled to these point like sources [7]. It is this field theoretic view-point that originally led to the estimate of the full microstate counting of the horizon from the dimension of the Hilbert space of CS theory by Kaul and Majumdar [7], using the machineries of topological quantum field theory from Witten's work [13] and Verlinde's formula from conformal field theory [27].

Step 2 begins with the imposition of a statistical mechanics. The black hole entropy calculation in LQG is based on the well-known Shannon entropy formula[2]

$$S = -\sum_{i=1}^{\Omega} p_i \ln p_i \tag{2}$$

where p_i is the probability of the i-th microstate and Ω is the total number of microstates of the system under consideration. Then considering that all the possible microstates can occur, a priori, with equal probability, one uses $p_i = 1/\Omega$ for all i in eq. (2) to arrive at the formula

$$S = \ln \Omega. \tag{3}$$

Since the estimate of this Ω is now known from the knowledge of the Hilbert space, the rest is just a mathematical procedure that leads to $S = (\lambda_0/\gamma)A$.

Now, one can easily make the observation that in Step 2, the computation of the entropy from the horizon microstates using eq. (2) inherently considers that the microstates are unbiased. This is possible only if these states *were* completely unaffected by any interaction with some external fields. On the other hand, in Step 1, we can see that the microstates of the horizon are described by a quantum CS theory coupled to point-like sources from bulk quantum geometry. There is a gravitational coupling or interaction between the horizon and the bulk. Hence, it seems quite logical to introduce some different statistical mechanics or rather an entropy formula more generalized than eq. (2) to take into account the effect of the coupling between the horizon and the bulk as a bias in the microstates. Now, the question is how should the microstates be biased. The answer has two physical views:

i) From the statistical mechanical viewpoint, the bias should be such that the entropy calculation from LQG leads to the BHAL.

ii) From the field theoretic viewpoint, the bias should be such that it increases with the strength of the coupling of the horizon microstates with the bulk geometry.

In this work we show that these two viewpoints complement each other quite naturally if we introduce a generalization of the Shannon entropy (henceforth to be called as q-entropy) to incorporate the effect of a bias in the microstates, use it to calculate the black hole entropy and demand it to yield the BHAL. As a consequence, we obtain the BHAL from the black hole microstates in LQG for arbitrary real positive values of γ. Nevertheless, at the end γ should have a fixed value that has to be determined by experimental means. Once we are able to do so, the parameter q will become a function of A which is physically well justified because of the following reason. The coupling strength between the horizon and the bulk is dependent on A as $k = A/4\pi\gamma$. Since q represents the effect of the bias created in the horizon microstates due to this coupling, it should also depend on A.

[2] We shall consider the Boltzmann constant to be unity.

3. The q-entropy

Originally, the idea behind the introduction of the notion of q-entropy was to incorporate, at the statistical mechanical level, the effect of a bias[3] in the probabilities of the microstates of the underlying quantum mechanical system [1] (also, see page 43 of [2]). The parameter q is called entropic index. In general we have $0 < p_i < 1$. Hence, $p_i^q > p_i$ for $q < 1$ and $p_i^q < p_i$ for $q > 1$. This implies $q < 1$ relatively enhances the rare events whose probabilities are close to zero and $q > 1$ relatively enhances the frequent events whose probabilities are close to unity. Intrigued by this fact, the q-entropy was postulated by Tsallis [1], which is given by

$$S_q = \frac{(1 - \sum_{i=1}^{\Omega} p_i^q)}{(q-1)}, \tag{4}$$

The parameter q is real and in the limit $q \to 1$ one recovers eq. (2). To mention, the related branch of statistical mechanics is known as non-extensive statistical mechanics (NESM)[4] due to its salient features [2]. For equal probability we have $p_i = 1/\Omega$ for all i. In this case eq. (4) reduces to

$$S_q = \ln_q \Omega \tag{5}$$

where $\ln_q x = (1 - x^{1-q})/(q-1)$ is called q-logarithm. The q-entropy for a spin sequence $(j_1, \cdots, j_N)$ can be calculated by putting $\Omega(j_1, \cdots, j_N) = \prod_{l=1}^{N}(2j_l + 1)$ in eq. (5). The expression comes out to be

$$S_q(j_1, \cdots, j_N) = \ln_q \prod_{l=1}^{N}(2j_l+1) = \frac{\left[\prod_{l=1}^{N}(2j_l+1)\right]^{1-q} - 1}{(1-q)} \tag{6}$$

If we consider $j_1 = \cdots = j_N = s$ (say), then eq. (6) reduces to the following form:

$$\begin{aligned} S_q^{(s)} &:= S_q(N \text{number of spin } s) \\ &= \frac{[(1+2s)^{(1-q)N} - 1]}{(1-q)} \end{aligned} \tag{7}$$

which was derived in [3]. This is the equation which we shall implement to calculate black hole entropy.

4. Black hole microstates in LQG

Now, let us briefly discuss the essential structures of the quantum geometry of black holes [6,7]. The quantum geometry of a cross-section of a black hole horizon in LQG is described by a topological two-sphere with defects, usually called punctures, carrying 'spin'[5] quantum numbers endowed by the edges of the spin network that represent the bulk quantum geometry [6]. Quantum area of the black hole with spin quantum numbers $j_1, \cdots, j_N$ on N punctures is given by $A_{qu} = 8\pi\gamma \sum_{l=1}^{N} \sqrt{j_l(j_l+1)}$ and the number of microstates is given by

$$\Omega(j_1, \cdots, j_N) = \prod_{l=1}^{N}(2j_l+1). \tag{8}$$

[3] The exact nature of bias in a quantum system can only come through the study of its dynamics. q-entropy is a way to incorporate that effect at the statistical mechanical level. This is the reason to put the word 'bias' within quotes in the abstract.

[4] The nomenclature 'non-extensive' is slightly misleading (see page 44 of [2]). However, we use it here as this branch of statistical mechanics is well known by this name.

[5] These 'spin' quantum numbers are not to be confused with particle spins. For an elaborate discussion see [14].