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Non-universal and universal aspects of the large scattering length limit

Gerald A. Miller

Department of Physics, University of Washington, Seattle, WA 98195-1560, United States

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ABSTRACT

The momentum density, $n(k)$ of interacting many-body Fermionic systems is studied (for $k > k_F$) using examples of several well-known two-body interaction models. It is shown that $n(k)$ can be approximated by a zero-range model for momenta k less than about $0.1/r_e$, where r_e the effective range. If the scattering length is large and one includes the effects of a fixed value of $r_e \neq 0$, $n(k)$ is almost universal for momenta k up to about $2/r_e$. However, $n(k)$ can not be approximated by a zero-range model for momenta k greater than about $1/(ar_e^2)^{1/3}$, where a is the scattering length, and if one wishes to maintain a sum rule that relates the energy of a two component Fermi-gas to an integral involving the density. We also show that the short separation distance, s, behavior of the pair density varies as s^6.

Interacting many-body Fermionic systems are copious in nature, with examples occurring in astrophysics, nuclear physics, condensed matter physics, and most recently in atomic systems. The development of trapping, cooling and magnetic resonance techniques for ultracold atoms allows the strength of the two-body interaction to be controlled experimentally [1–3]. At low relative energies this strength is characterized by the scattering length, a, which can be much, much larger than the range of the two-body interactions, R.

Such systems are of interest to understanding the nucleon–nucleon interaction, which is characterized at low energies by scattering lengths of magnitude much larger than the effective range. Indeed, the limit of $a \to \infty$, $R \to 0$, defined as the unitary limit, has been used as a benchmark problem for nuclear many-body physics. See for example, G.F Bertsch in Ref. [4] and for example [5, 6].

If a/R approaches infinity, the system is expected to have universal properties that are determined only by the scattering length. Tan [7–9] derived universal relations between diverse properties of any arbitrary system consisting of fermions in two spin states with a large scattering length. These relations include the coefficient of the $1/k^4$ tail of the momentum distribution [7], a decomposition of the energy into terms that insensitive to short distances [7], an expression for the local pair density [7], and various other properties of interacting many fermion systems [8–12]. Tan's derivations start with the assumption that the interaction between constituents is a zero-range pseudopotential. Brataan and Platter [13] confirmed these relations using a zero-range interaction, renormalized by cutting off the intermediate momentum integrals at high values of the momentum. But the range of interaction is never 0 for any physical system even though the ratio R/a can be made exceedingly.

The aim of the present epistle is to explore the consequences of the non-zero range of interaction. We study how the effects of the non-zero value of the effective range influence the relative two-fermion wave function of the interacting two-bodies. The square of the momentum–space wave function, $\widetilde{\phi}(k)$ determines the shape of the system's momentum distribution, $n(k) = \widetilde{\phi}^2(k)$ for $k > k_F$, the Fermi momentum. Recent studies that examine the non-zero nature of the effective range include [14–16]. Ours focuses on exhibiting the most relevant features of the two-body system.

We note that the universal relations all involve a property of the system that depends on the contact density, $\mathcal{C}(\mathbf{R})$, which is the local density of interacting pairs. Its integral over volume is denoted the contact, C [7], said to be a measure of the number of atom pairs with large (but not too large) relative momentum [17]. Recent works that apply the contact formalism to nuclear physics include Refs. [18–21].

At low relative energies the s-wave scattering phase shift δ can be expressed in terms of the effective range expansion:

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2}r_e k^2 + \cdots, \tag{1}$$

where k is the relative momentum ($\hbar^2/m$ is taken as unity by convention), a is the scattering length and r_e is the effective range. The effective range is expected to be of the order of range of the two-body interactions that govern the system, but a can be much larger in magnitude. The present analysis is concerned with cases for

E-mail address: miller@phys.washington.edu.

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which $a > 0$. If $a \gg r_e > 0$, the S-matrix element $e^{2i\delta(k)}$ has a pole on the positive imaginary axis. This pole corresponds to the energy of a bound state of very small binding energy, $B = 1/a^2 + \frac{r_e}{a^3} + \cdots$, in units with $\hbar^2/m = 1$, with m as twice the reduced mass of the interacting pair.

With a zero range interaction, the resulting wave function is simply $\phi_0(r) = u_0(r)/r = \sqrt{2/(4\pi a)}\exp(-r/a)/r$, with the momentum space version, $\widetilde{\phi}_0(k) = \sqrt{8\pi a}a/(k^2a^2+1)$, and $n_0 = \widetilde{\phi}_0^2$. This function is the source of the claimed $1/k^4$ behavior of the density. The range of validity of these expressions is said [7,17] to be

$$1/a \ll k \ll 1/r_e. \tag{2}$$

If r_e is taken to 0, then the upper limit would be infinite.

But $r_e \neq 0$ for all physical systems, so that other momentum scales may enter. For example, consider the effective range expansion of Eq. (1). In the large a limit, the first term is very small. Thus the second term can be as large as or much larger than the second term for relatively small values of k. For example, if $k = \sqrt{\frac{2}{ar_e}}$, the second term of Eq. (1) provides a 100% correction to the first term. An effect of that size cannot be ignored. The calculations will show that other momentum scales smaller than $1/r_e$ are important.

We study the influence of the non-zero range of the interaction as manifest by the difference between $n(k)$ and $n_0(k)$. The physics of the interior must matter because, $(\nabla^2 + \frac{1}{a^2})e^{-r/a}/r = -4\pi\delta(\mathbf{r})$, so that the function $e^{-r/a}/r$ is not a solution of the Schroedinger equation in the usual sense. The region with $r \le r_e$ matters, r_e cannot be set to 0, and the range of validity of the $1/k^4$ behavior must be limited.

Our approach is to analyze four simple available models of the interaction that have the same non-zero effective range, and then to draw some general conclusions. Starting with an attractive, square-well, (S), potential of depth V_0 and range R is useful. The bound state wave function is given by

$$\phi_S(r) = \frac{N_S}{r}\Big[\frac{\sin Kr}{\sin KR}\theta(R-r) + e^{-(r-R)/a}\theta(r-R)\Big], \tag{3}$$

where N_S is a normalization factor, and $K = \sqrt{V_0 - B}$. For very large scattering lengths KR is slightly larger than $\pi/2$, and the effective range r_e is very close to R, the range of the interaction. The momentum space wave function is the Fourier transform:

$$\widetilde{\phi}_S(k) = \bar{N}_S \frac{\left(\frac{\sin(kR)}{ak} + \cos(kR)\right)}{\left(K^2-k^2\right)\left(1+a^2k^2\right)}. \tag{4}$$

Another simple model is the Hulthein wave function, (H). The bound state wave function is given by

$$\phi_H(r) = \frac{\sqrt{\beta(\beta+\alpha)}}{\beta-\alpha}\sqrt{\frac{2\alpha}{4\pi}}\frac{1}{r}(e^{-\alpha r} - e^{-\beta r}), \tag{5}$$

with $\beta \gg> \alpha$, $B = \alpha^2$. In the large a/r_e limit, $\alpha = 1/a$. The momentum space wave function is given by

$$\widetilde{\phi}_H(k) = 2\sqrt{2\pi}\sqrt{\alpha\beta(\alpha+\beta)^3}\frac{1}{\alpha^2+k^2}\frac{1}{\beta^2+k^2}. \tag{6}$$

The surface delta (SD) function potential $V(r) \propto \delta(r-R)$ is also easily analyzed. The coordinate space wave function is given by

$$\phi_{SD}(r) = \frac{N_{SD}}{r}\Big(\frac{\sinh(r/a)}{\sinh(R/a)}\theta(R-r) + e^{-(r-R)/a}\theta(r-R)\Big), \tag{7}$$

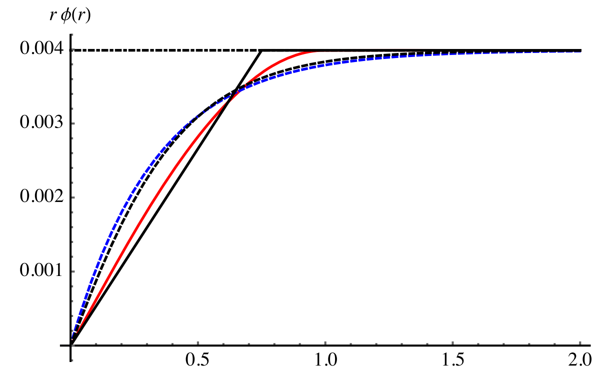

Fig. 1. (Color online.) Comparison of $r\phi_S(r)$ (red) solid, $r\phi_H(r)$ (blue) dashed, and $r\phi_0(r)$ (black) dot-dashed, $r\phi_{SD}(r)$ (black) solid, and $r\phi_E(r)$ (black) dashed for $a/R = 10,000$. Dimensionless units are used (see text).

if $B = 1/a^2$. The momentum space wave function is given by

$$\widetilde{\phi}_{SD}(k) = \tilde{N}_{SD}\frac{1}{1+(ka)^2}\frac{\sin(kR)}{kR}. \tag{8}$$

The exponential potential (E), $V(r) = -V_{0E}e^{-\mu r}$ provides another well-studied example. A bound state wave function is given by

$$r\phi_E(r) = N_E J_{2\sigma/\mu}(2\sqrt{V_{0E}}/\mu\, e^{-\mu r/2}), \tag{9}$$

with $V_0 > 0$, and the value of σ determined by the condition $J_{2\sigma/\mu}(2\sqrt{2V_{0E}/\mu}) = 0$, $B = -\sigma^2$, and $1/\sigma = a$ in the large scattering length limit. A useful form of the momentum–space wave function is obtained by using the power-series expansion for the Bessel function. The large scattering length result is given by

$$\widetilde{\phi}_E(k) = \tilde{N}_E\sum_{j=0}^{\infty}(\frac{V_{0E}}{\mu^2})^{j+1/(a\mu)} \times \frac{(-1)^j}{j!\Gamma(j+1+2/(\mu a))}\frac{1}{k^2+(1/a+\mu j)^2}. \tag{10}$$

This function can be seen as a generalization of the Hulthein wave function.

Five coordinate space wave functions $(0, S, H, SD, E)$ are compared in Fig. 1. As noted The effective range is chosen to be unity in the natural length unit of the system. The examples shown here use the very large ratio for $a/R = 10000$ to ensure that any non-universal features do not arise from an insufficiently large scattering length. The values $V_{0E} = 18.73886$, $\mu = 3.6$ are used to obtain same values of a, r_e as for the other potentials. The effective range is chosen to be unity in the natural length unit of the system. If the binding energy is very small, the effective range can be computed using the bound state wave function. The relevant expression is (for $B = -1/a^2$)

$$r_e = 2\int dr(e^{-2r/a} - u_{ER}^2(r)), \tag{11}$$

where $u_{ER}(r) = r\phi(r)$ normalized so that its asymptotic form is $e^{-r/a}$. This expression is taken from the usual effective range expansion [22], but using $k = i/a$: the energy is taken to approach 0 from negative values. Any differences from the approach using positive energy are accounted for by higher order terms in the effective range expansion, which are significant for small values of k. For the square well, $R \approx 1$. For the Hulthein wave function one finds: