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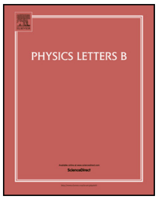

The zeroth law in quasi-homogeneous thermodynamics and black holes

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ABSTRACT

Motivated by black holes thermodynamics, we consider the zeroth law of thermodynamics for systems whose entropy is a quasi-homogeneous function of the extensive variables. We show that the generalized Gibbs–Duhem identity and the Maxwell construction for phase coexistence based on the standard zeroth law are incompatible in this case. We argue that the generalized Gibbs–Duhem identity suggests a revision of the zeroth law which in turns permits to reconsider Maxwell's construction in analogy with the standard case. The physical feasibility of our proposal is considered in the particular case of black holes.

1. Introduction

The thermodynamics of black holes contains several peculiarities in contrast to standard thermodynamics. One example is the different scaling behaviour when rescaling the thermodynamic variables. This can be directly verified noting that the entropy of a black hole is – in general – a quasi-homogeneous function of the extensive thermodynamic quantities describing the system [1], and its scaling behavior is dictated by the Smarr relation. Such systems are generically called *quasi-homogeneous*. As a consequence, it is usually recognized that using the formalism of homogeneous thermodynamics in the case of black holes is not fully justified and that a modification of the thermodynamic laws for systems with quasi-homogeneous entropy is called for [1].

It has been established that in systems where entropy and energy are not additive the standard way to define equilibrium has to be adjusted and, in such case, the thermodynamic temperature may not be the correct parameter to be equated at equilibrium [2–9]. In spite of this, it has been repeatedly argued in favor of the existence of first order phase transitions – i.e., coexistence processes – within the framework of black hole thermodynamics. Such arguments are based on the analogy with the van der Waals (vdW) phase diagram and use the Maxwell equal area law to find the coexistence curve *as if* the system was homogeneous (see e.g. [1,10–19] and the references therein).

In this work we consider systems whose entropy is a quasi-homogeneous function of the extensive variables and show that Maxwell's equal area law – based on the definition of thermodynamic equilibrium for homogeneous systems (cf. [20] and the discussion in Section 4.3 in [21]) – is inconsistent with the generalized Gibbs–Duhem (GGD) identity that must hold in such cases [22,23]. We show that this situation can be remedied introducing a new set of the variables defining equilibrium. Based on these generalized variables, we propose a definition of thermodynamic equilibrium originating from the GGD identity and we demonstrate that such revision is essential in Maxwell's construction for phase coexistence. It is worth mentioning that our *generalized zeroth law* reduces to the standard definition for homogeneous systems of degree one.

To illustrate our proposal we discuss two relevant cases: on the one hand, we show that for the Schwarzschild black hole the *new* temperature characterizing equilibrium is constant, i.e. it does not depend on its mass M. This coincides (up to a constant factor) with the result in [24], where such parameter is obtained using a gener-

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alized zeroth law for non-extensive statistical mechanics developed in [6]. This proves that, at least in the Schwarzschild case, there is a consistency between different approaches. On the other hand, we consider the first order phase transition in the Kerr–Anti de Sitter (Kerr–AdS) family of black holes and show that the Maxwell construction as applied in the literature leads to a violation of the GGD. Using the new generalized intensive parameters and according to our definition of thermodynamic equilibrium, such transition seems to disappear. Given the importance of this example in the context of the AdS/CFT correspondence, we believe that this can be relevant for future investigations.

This paper is structured as follows. In Section 2 we review the thermodynamics of quasi-homogeneous systems as developed in [22,23]. In Section 3 we point out the aforementioned mathematical inconsistency between Maxwell's construction based on the standard zeroth law of thermodynamics and the Gibbs–Duhem relation in the case of quasi-homogeneous entropy, and continue by proposing a generalized form of the zeroth law, which is consistent with the corresponding GGD relation. To illustrate the new form of the zeroth law, we consider the examples of Schwarzschild and Kerr–AdS black holes in Section 4, before we conclude in Section 5. Throughout this work we use Planck units, in which $c = G = \hbar = k_B = 1$.

2. Quasi-homogeneous thermodynamics

In this section we briefly review some results of the thermodynamics of quasi-homogeneous systems obtained in [22,23]. Let us start by recalling some definitions. Unless otherwise stated, we will not use Einstein's sum convention.

Definition 2.1 *(Quasi-homogeneous function).* Let $r, \lambda \in \mathbb{R}$, $\lambda \neq 0$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$. A function w of a set of variables $\{q^i\}_{i=1}^n$ is said to be *quasi-homogeneous of degree r and type β* if

$$w(\lambda^{\beta_1} q^1, \ldots, \lambda^{\beta_n} q^n) = \lambda^r w(q^1, \ldots, q^n)\,. \tag{1}$$

The particular case where $\beta_i = 1$ for every value of i yields the standard scaling relation of homogeneous functions of degree r, i.e.

$$w(\lambda q^1, \ldots, \lambda q^n) = \lambda^r w(q^1, \ldots, q^n)\,. \tag{2}$$

In the following we will use S instead of w, because the function of interest in thermodynamics is the entropy. The variables $\{q^i\}_{i=1}^n$ are the extensive variables of the system, such as internal energy U, volume V or number of particles N. In standard thermodynamics of extensive systems the entropy is a homogeneous function of degree one of the extensive variables, i.e.,

$$S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)\,, \tag{3}$$

while in black holes thermodynamics the entropy is a quasi-homogeneous function as in Definition 2.1.

Proposition 2.1. *Let $S = S(q^1, \ldots, q^n)$ be a quasi-homogeneous function of degree r and type β. Then, the conjugate variables to the q^i, defined by*

$$p_i\left(q^j\right) \equiv \frac{\partial}{\partial q^i} S\left(q^j\right)\,, \tag{4}$$

are quasi-homogeneous functions of degree $r - \beta_i$ for every value of i.

Proof.

$$\begin{aligned} p_i\left(\lambda^{\beta_j} q^j\right) &= \frac{\partial}{\partial(\lambda^{\beta_i} q^i)} S\left(\lambda^{\beta_j} q^j\right) = \frac{1}{\lambda^{\beta_i}} \frac{\partial}{\partial q^i}\left[\lambda^r S\left(q^j\right)\right] \\ &= \lambda^{r-\beta_i} \frac{\partial}{\partial q^i} S(q^j)\,. \end{aligned} \tag{5}$$

Therefore

$$p_i\left(\lambda^{\beta_j} q^j\right) = \lambda^{r-\beta_i} p_i(q^j)\,. \quad \square \tag{6}$$

Note that if S is homogeneous of degree $r = 1$ [cf. equation (3) above], then the conjugate variables p_i are homogeneous functions of degree 0, i.e. $p_i(\lambda q^j) = p_i(q^j)$, i.e., they do not change when the system is re-scaled. Only in this case, the conjugate variables are *intensive* and we recover the usual thermodynamic quantities, e.g. $1/T$, p/T, μ/T. In all other cases we shall refer to the conjugate variables p_i as the *would-be intensive* quantities, as in [22,23].

Proposition 2.2 *(Euler's Theorem). Let $S = S(q^1, \ldots, q^n)$ be a quasi-homogeneous function of degree r and type β. Then*

$$r S(q^j) = \sum_{i=1}^n \beta_i \left[q^i p_i(q^j)\right]. \tag{7}$$

Proof. Consider the derivative of $S(\lambda^{\beta_j} q^j)$ with respect to the scaling parameter λ. On the one hand, since S is a quasi-homogeneous function of degree r and type β, we have

$$\frac{\partial}{\partial \lambda} S(\lambda^{\beta_j} q^j) = \frac{\partial}{\partial \lambda}\left[\lambda^r S(q^j)\right] = r\lambda^{r-1} S(q^j)\,. \tag{8}$$

On the other hand, a direct calculation yields

$$\begin{aligned} \frac{\partial}{\partial \lambda} S(\lambda^{\beta_j} q^j) &= \sum_{i=1}^n \frac{\partial S(\lambda^{\beta_j} q^j)}{\partial(\lambda^{\beta_i} q^i)} \frac{\partial(\lambda^{\beta_i} q^i)}{\partial \lambda} \\ &= \sum_{i=1}^n \frac{\partial S(\lambda^{\beta_j} q^j)}{\partial(\lambda^{\beta_i} q^i)} \left(\beta_i \lambda^{\beta_i - 1} q^i\right) \\ &= \sum_{i=1}^n \left(\beta_i \lambda^{r-1} q^i\right) p_i(q^j), \end{aligned} \tag{9}$$

where the last equality follows from Definition (4) and Eqs. (5) and (6). Thus, combining the results of (8) and (9), Eq. (7) is obtained. $\square$

In standard thermodynamics the above result reduces to the well-known identity for the entropy,

$$S = \frac{1}{T} U - \frac{p}{T} V + \frac{\mu}{T} N\,. \tag{10}$$

With Proposition 2.2, we can write a GGD relation for the case of quasi-homogeneous thermodynamic systems.

Proposition 2.3 *(Generalized Gibbs–Duhem identity). Let $S(q^1, \ldots, q^n)$ be a quasi-homogeneous function of degree r and type β and let $\{p_i\}_{i=1}^n$ be the set of conjugate variables [cf. equation (4)]. Then,*

$$\sum_{i=1}^n \left[(\beta_i - r)\, p_i(q^j) \mathrm{d}q^i + \beta_i q^i \mathrm{d}p_i(q^j)\right] = 0\,. \tag{11}$$

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