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The two-loop soft current in dimensional regularization

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ABSTRACT

The soft current describes the factorization behavior of quantum chromodynamics (QCD) scattering amplitudes in the limit of vanishing energy of one of the external partons. It is process-independent and can be expanded in a perturbative series in the coupling constant. To all orders in the dimensional regularization parameter, we compute the two-loop correction to the soft current for processes involving two hard partons.

In the limit where one or more massless partons are unresolved, amplitudes in quantum field theory factorize into lower-point amplitudes with the unresolved partons removed, times a universal function that is independent of the details of the hard interaction and describes the emission of the unresolved particles. This factorization property has multiple implications both for the formal study of scattering in quantum field theory and for the phenomenology of scattering processes at high-energy particle colliders. On the formal side, several conjectures on the high-order behavior of perturbation theory and on the all-order structure of the scattering matrix are formulated based on insights gained from the limiting behavior in unresolved limits. In precision applications of perturbation theory to collider phenomenology, systematic expansions around unresolved limits allow to approximate or reconstruct higher-order coefficients in an elegant and computationally efficient manner.

In particular, the emission of a soft gluon is entirely described by the so-called *soft current*, an operator in color space that encapsulates all the information on the soft emission. In this Letter we report on the computation of the two-loop soft current in QCD for the emission of a single soft gluon from an amplitude involving two hard colored particles.

Let us consider the amplitude $|\mathcal{M}(q, p_1, \dots, p_n)\rangle$ (as vector in color space) for a gluon with momentum q in association with n colored particles with momenta p_i, $i = 1, \dots, n$, transforming in some irreducible representation of $SU(N)$. The amplitude may depend furthermore on an arbitrary number of colorless particles. In the soft limit where the energy of the gluon vanishes, the amplitude factorizes according to

$$\langle a|\mathcal{M}(q, p_1, \dots, p_n)\rangle \simeq \varepsilon^\mu(q) J^a_\mu(q)|\mathcal{M}(p_1, \dots, p_n)\rangle, \tag{1}$$

where a denotes the adjoint color index of the soft gluon and $\varepsilon^\mu(q)$ its polarization vector, and the '$\simeq$' sign indicates that the equality only holds up to the leading term in the expansion in the soft gluon momentum. Eq. (1) defines the (unrenormalized) soft current $J^a_\mu(q)$, which describes the emission of a soft gluon. Both the soft current and the amplitude admit a perturbative expansion,

$$|\mathcal{M}(p_1, \dots, p_n)\rangle = \sum_{\ell=0}^{\infty} |\mathcal{M}^{(\ell)}(p_1, \dots, p_n)\rangle,$$

$$J^a_\mu(q) = g_S \mu^\epsilon \sum_{\ell=0}^{\infty} (g_S \mu^\epsilon)^{2\ell} J^{a(\ell)}_\mu(q). \tag{2}$$

We work in $D = 4 - 2\epsilon$ dimensions and g_S denotes the bare QCD coupling constant and μ is the scale introduced by dimensional regularization. The tree-level soft current is given by the well-known eikonal factor,

$$J^{a(0)}_\mu(q) = \sum_{i=1}^{n} T^a_i \frac{p_{i\mu}}{p_i \cdot q}, \tag{3}$$

where T^a_i denote the generators of $SU(N)$ of the representation of parton i. The one-loop correction was computed in Ref. [1],

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$$J_\mu^{a(1)}(q) = -\frac{S_\epsilon}{16\pi^2}\frac{1}{\epsilon^2}\Gamma(1-\epsilon)\Gamma(1+\epsilon) i f^{abc} \times \sum_{i\neq j} T_i^b T_j^c \left(\frac{p_{i\mu}}{p_i\cdot q} - \frac{p_{j\mu}}{p_j\cdot q}\right)\left[\frac{(-s_{ij})}{(-s_{iq})(-s_{qj})}\right]^\epsilon, \quad (4)$$

with $s_{ij} = 2p_i \cdot p_j + i0$ and

$$S_\epsilon = (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}. \quad (5)$$

We emphasize that the soft current is not a scalar quantity, but an operator in color space, i.e., it acts non-trivially on the color indices of the hard amplitude. While at tree and one-loop level, all color operators involve at most two hard partons, starting from two loops new color structures may appear that connect up to three hard partons [2]. Non-trivial contributions from these new color structures to the soft current cannot be excluded.

If we focus on processes with only two hard colored particles, the color structure of the soft current drastically simplifies. Indeed, color conservation in the hard amplitude implies that the two hard partons must transform in complex conjugate representations. The color structure is then most conveniently described by color-ordered helicity amplitudes,

$$\big|\mathcal{M}(q,p_1,p_2)\big\rangle = T^a_{i_1 i_2} A(q,p_1,p_2), \quad (6)$$

where the color-ordered amplitude $A(q,p_1,p_2)$ depends on the helicities and momenta of the colored particles, but not on their color. In the limit where the gluon becomes soft, the color-ordered amplitude factorizes,

$$A(q,p_1,p_2) \simeq g_S \mu^\epsilon \mathcal{S}_\pm(q) r_{soft}(q) A(p_1,p_2). \quad (7)$$

The helicity dependence of the soft emission is entirely encoded into the tree-level soft function,

$$\mathcal{S}_+(q) = \sqrt{2}\frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle} \quad \text{and} \quad \mathcal{S}_-(q) = -\sqrt{2}\frac{[12]}{[1q][q2]}. \quad (8)$$

Here $\langle ij\rangle$ and $[ij]$ denote the usual spinor products, related to the Mandelstam invariants by $s_{ij} = \langle ij\rangle[ji]$. Quantum corrections to the soft emission are helicity-independent and expressed in the scalar function

$$r_{soft}(q) = 1 + \sum_{\ell=1}^\infty \left\{ g_S^2 \mu^{2\epsilon}\frac{S_\epsilon}{16\pi^2}\left[\frac{(-s_{12})}{(-s_{1q})(-s_{q2})}\right]^\epsilon\right\}^\ell r_{soft}^{(\ell)}. \quad (9)$$

The coefficients $r_{soft}^{(\ell)}$ are related to the soft current by

$$J_\mu^{a(0)}(q) J_a^{\mu(\ell)}(q) = -4C_i\left(\frac{S_\epsilon}{16\pi^2}\right)^\ell \left[\frac{(-s_{12})}{(-s_{1q})(-s_{q2})}\right]^{1+\ell\epsilon} r_{soft}^{(\ell)}, \quad (10)$$

where C_i is the Casimir operator in the representation of the two hard partons. Comparing Eq. (10) to Eq. (4) and performing the color algebra, we immediately see that the one-loop coefficient is given by

$$r_{soft}^{(1)} = -N\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\epsilon^2}. \quad (11)$$

The two-loop coefficient was computed in Ref. [3] up to $\mathcal{O}(\epsilon^0)$ by considering the soft limit of the two-loop amplitudes for $\gamma^* \to Q\bar{Q}g$ [4] and $H \to 3$ partons [5]. For applications in precision calculations, the soft current is to be integrated over the soft phase space (giving rise to a double pole in the regularization parameter), and is therefore required to $\mathcal{O}(\epsilon^2)$.

The two-loop coefficient $r_{soft}^{(2)}$ can be extracted from a given two-loop amplitude involving two hard partons and a gluon. We focus on the D-dimensional two-loop amplitude for $\gamma^* \to Q\bar{Q}g$, interfered with the tree-level amplitude and summed over colors and spins [6]. The matrix element is a function of the lightlike momenta p_1, p_2 and q of the quark pair and the gluon. In the limit where the gluon becomes soft, it factorizes according to,

$$\big\langle\mathcal{M}_3^{(0)}\big|\mathcal{M}_3^{(2)}\big\rangle \simeq -g_S^2\mu^{2\epsilon}\sum_{\ell=0}^2 \big(g_S^2\mu^{2\epsilon}\big)^\ell \big\langle\mathcal{M}_2^{(0)}\big|J_\mu^{a(0)} J_a^{\mu(\ell)}\big|\mathcal{M}_2^{(2-\ell)}\big\rangle. \quad (12)$$

The two-loop coefficient $r_{soft}^{(2)}$ can then directly be extracted by expanding in the soft gluon momentum.

If we denote the virtuality of the photon by $Q^2 = (p_1+p_2+q)^2$, then (up to some overall power of Q^2) the matrix element $\langle\mathcal{M}_3^{(0)}|\mathcal{M}_3^{(2)}\rangle$ can only depend on the Lorentz-invariant dimensionless ratios

$$x = \frac{s_{12}}{Q^2}, \qquad y = \frac{s_{1q}}{Q^2}, \qquad z = \frac{s_{2q}}{Q^2}, \quad (13)$$

subject to the constraints

$$x+y+z = 1 \quad \text{and} \quad 0 < x,y,z < 1. \quad (14)$$

Without loss of generality, we set $Q^2 = 1$ in the following. The soft limit is then approached when both y and z tend to zero at the same rate. Our goal is thus to expand the matrix element into a power series in y and z while keeping the dependence of the coefficients on the dimensional regulator ϵ exact. The leading term of the expansion then corresponds to the right-hand side of Eq. (12).

The two-loop amplitude for $\gamma^* \to Q\bar{Q}g$ can be written as a linear combination of scalar four-point master integrals with one external massive leg [7,8]. In the following we denote the master integrals collectively by $F_i(y,z;\epsilon)$. The master integrals themselves satisfy a system of coupled differential equations that can schematically be written as

$$\frac{\partial}{\partial y}F_i(y,z;\epsilon) = A_{ij}^y(y,z;\epsilon)F_j(y,z;\epsilon),$$
$$\frac{\partial}{\partial z}F_i(y,z;\epsilon) = A_{ij}^z(y,z;\epsilon)F_j(y,z;\epsilon), \quad (15)$$

where $A_{ij}^k(y,z;\epsilon)$, $k \in \{y,z\}$, are rational functions of y, z and ϵ. Solutions to Eqs. (15) valid to all orders in ϵ are only available in a few special cases [7]. Laurent expansions in ϵ were obtained for all master integrals up to $\mathcal{O}(\epsilon^0)$ in terms of harmonic polylogarithms and their two-dimensional generalization [8]. These results yield the two-loop amplitude for $\gamma^* \to Q\bar{Q}g$ up to $\mathcal{O}(\epsilon^0)$. Expanding the two-dimensional harmonic polylogarithms as power series in y and z immediately reproduces the known result for $r_{soft}^{(2)}$ up to $\mathcal{O}(\epsilon^0)$ [3].

To obtain the two-loop coefficient $r_{soft}^{(2)}$ to all orders in ϵ, we return to the differential equations (15) and construct for each master integral a power series solution in y and z close to the origin $(y,z) = (0,0)$ in the (y,z) plane. The differential equations may, however, have poles whenever y or z vanish, translating into branching points for the master integrals starting from points where one of the two expansion parameters is zero. In other words, the solutions to Eq. (15) are not meromorphic in a neighborhood of the origin of the (y,z) plane, and so we cannot make a simple Laurent series ansatz in y and z for the master integrals. The correct ansatz for each master integral rather takes the form

$$F_i(y,z;\epsilon) = \sum_{m,n=0}^2 y^{-m\epsilon}z^{-n\epsilon}f_{i,mn}(y,z;\epsilon), \quad (16)$$