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Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions

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ABSTRACT

Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams are studied through a finite element approach. The exact shape functions for uniform homogeneous Timoshenko beam elements are used to formulate the proposed element. The accuracy of the present element is considerably improved by considering the exact variations of cross-sectional profile and mechanical properties in the evaluation of the structural matrices. Carrying out several numerical examples, the convergence of the method is verified and the effects of taper ratio, elastic constraint, attached mass and the material non-homogeneity on the natural frequencies and critical buckling load are investigated.

1. Introduction

Functionally graded (FG) materials have received great interest from engineers and researchers due to their specific characteristics such as high stiffness and thermal resistance. FG materials are formed by varying percentage content of materials in any desired spatial direction; as a result, the mechanical properties of the new material would show gradation in that direction which could be formulated using different theories among which the most renowned ones are exponential [1] and power laws [2]. Reviewing the literature, it is understood that most of the previous works on FG beams have considered the gradation of material properties in thickwise direction [3–5] and relatively few researchers [6–16] have studied the structural behavior of those types of FG beams whose material properties vary in lengthwise direction. Recent works on axially FG beams have considered Euler–Bernoulli beam theory (EBT). It is well understood that EBT [17–21] neglects the effects of shear deformation and rotary inertia while Timoshenko beam theory (TBT) takes these effects into account. Therefore, long slender beams could be efficiently modeled by EBT since the flexural behavior is dominant while application of EBT leads to great inaccuracy in modeling of short thick beams, where the shear deformations are of more significance. Moreover, it is well known that TBT presents a more realistic model of the beam in determination of the higher modes of vibration.

Non-prismatic beams, i.e. those beams whose cross-sectional profile changes gradually or abruptly along their length are of great importance in different fields of engineering due to their ability in meeting the architectural and aesthetical needs and optimizing the weight and strength of the structure. This particular advantage of non-prismatic members, i.e. optimization of weight and strength, plays a very important role in construction and performance of aerospace structures. The key problem in analysis of tapered Timoshenko beams is the presence of variable coefficients in the governing differential equations introduced by variable cross-sectional area and moment of inertia. Due to this problem, there are closed-form solutions for neither free vibration nor stability of Timoshenko beams with an arbitrarily variable cross-section. Thus, numerical methods have been used among which finite element method has gained a more prominent position [22–25]. Analysis of tapered Timoshenko beams with arbitrary distributions of material properties along the beam axis are more complicated than the tapered Timoshenko beams since the variable material properties, in addition to the previously mentioned variable coefficients, show up in the governing differential equations. There are relatively few works [26–28] in the literature on the axially FG tapered Timoshenko beams. Tong et al. [26] modeled a tapered non-homogeneous Timoshenko beam as an assemblage of several uniform homogeneous segments, but they used their method to study the vibration characteristics of tapered Timoshenko beams with constant material properties. Sorrentino et al. [27] studied the effects of generalized damping on the non-homogeneous Timoshenko

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beams. Sklyar and Szkibiel [28] studied the controllability of a slowly rotating non-homogeneous Timoshenko beam.

To the authors' best knowledge, the stability analysis of axially FG tapered Timoshenko beams has not been performed previously and there is still a gap in the literature on the structural analysis of these beams and the effects of material non-homogeneity and taper ratio on their structural behavior. In this paper, finite element method is applied to study the free vibration and stability of these structural members. The most important part of a finite element formulation is the selection of shape functions, i.e. functions which interpolate the displacement field within an element. In this paper, the exact shape functions of a uniform homogeneous Timoshenko beam element are used to formulate the axially FG tapered Timoshenko beam element. This set of shape functions will propose an inaccuracy in the results; however this inaccuracy is lessened by considering the exact variation of cross-sectional profile and the mechanical properties in evaluation of the structural matrices. The present element could be used for modeling Timoshenko beams with any type of cross-sectional variations and gradations of material properties along the beam element; hence it could be used for most of engineering applications dealing with such beams.

The objective of the paper is to introduce a beam element which could be used to fill the present gap in the literature on axially FG tapered Timoshenko beams and present an insight into the structural behavior of these structural members. In the following, the structural model is firstly explained and afterwards the finite element formulations are presented. Finally numerical examples are provided to check the competency of the proposed element.

2. Structural model

Following the Timoshenko beam theory, the axial and transverse displacement fields are respectively given as

$$U(x,y,z,t) = -z\theta(x,t) \tag{1a}$$
$$W(x,y,z,t) = w(x,t) \tag{1b}$$

where x, y and z are the spatial coordinates as shown in Fig. 1, t is time, θ and w are respectively the bending rotation and the transverse displacement. Assuming that the beam element is subject to a constant compressive load P, the axial and shear strains are respectively obtained using Eq. (1) as,

$$\varepsilon_{xx} = -z\frac{\partial\theta}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2a}$$
$$\gamma_{xz} = \frac{\partial w}{\partial x} - \theta \tag{2b}$$

The stresses could be readily given in terms of strains as

$$\sigma_{xx} = E(x)\varepsilon_{xx} \tag{3a}$$
$$\tau_{xz} = G(x)\gamma_{xz} \tag{3b}$$

where σ_{xx} and τ_{xz} are respectively the axial and shear stresses, E and G are respectively the Young's modulus and the shear modulus which are both functions of longitudinal coordinate, x, to account

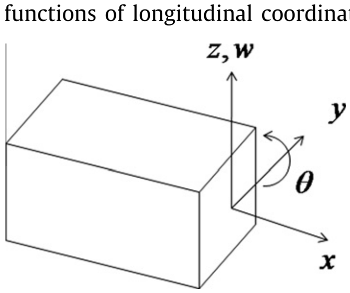

Fig. 1. Spatial coordinate system and the displacement field.

for the axial gradation of the material properties. Moreover, the strain energy and kinetic energy of the beam element could be written respectively as

$$U = \frac{1}{2}\int_0^l \int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})\,dA\,dx \tag{4a}$$
$$T = \frac{1}{2}\int_0^l \int_A \rho(x)(\dot{U}^2 + \dot{W}^2)\,dA\,dx \tag{4b}$$

Here the dot $(\cdot)$ stands for differentiation with respect to time, l is the element length, A is the cross-sectional area, ρ is the mass density which is a function of longitudinal coordinate due to the lengthwise gradation of the material. Applying Hamilton's principle [29], the following governing differential equations are obtained

$$\frac{\partial}{\partial x}\left(E(x)I(x)\frac{\partial\theta}{\partial x}\right) + \kappa G(x)A(x)\left(\frac{\partial w}{\partial x} - \theta\right) - \rho(x)I(x)\frac{\partial^2\theta}{\partial t^2} = 0 \tag{5a}$$

$$\frac{\partial}{\partial x}\left[\kappa G(x)A(x)\left(\frac{\partial w}{\partial x} - \theta\right)\right] - \frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right) - \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} = 0 \tag{5b}$$

Here I is the moment of inertia and κ is the shear correction factor which depends on the shape of the cross-section. Assuming sinusoidal variations for transverse displacement and bending rotation with circular frequency ω and using Eq. (5), the governing differential equations for free lateral vibration are attained as

$$\frac{d}{dx}\left(E(x)I(x)\frac{d\theta}{dx}\right) + \kappa G(x)A(x)\left(\frac{dw}{dx} - \theta\right) + \rho(x)I(x)\omega^2\theta = 0 \tag{6a}$$

$$\frac{d}{dx}\left[\kappa G(x)A(x)\left(\frac{dw}{dx} - \theta\right)\right] - \frac{d}{dx}\left(P\frac{dw}{dx}\right) + \rho(x)A(x)\omega^2 w = 0 \tag{6b}$$

It is well known that the transverse natural frequency vanishes when the axial compressive load equals critical load (P_{cr}); that is, ω is set to zero in stability analysis. Therefore, the following equation is derived directly from Eq. (6) for determination of critical buckling load.

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d\theta}{dx}\right) + P_{cr}\frac{d}{dx}\left[\theta - \frac{1}{\kappa G(x)A(x)}\frac{d}{dx}\left(E(x)I(x)\frac{d\theta}{dx}\right)\right] = 0 \tag{7}$$

3. Finite element formulation

As it was previously mentioned, Eqs. (6) and (7) do not have closed-form solutions due to the variable coefficients, thus the exact shape functions could not be determined. In this paper, the exact shape functions of homogeneous and isotropic uniform Timoshenko beams are used. These shape functions are given as [29].

Transverse Shape Functions:

$$\psi_1 = \frac{1}{\mu}[\mu - 12\Lambda\eta - (3-2\eta)\eta^2], \quad \psi_2 = \frac{l}{\mu}[\eta(1-\eta)^2 + 6\Lambda\eta(1-\eta)],$$

$$\psi_3 = \frac{1}{\mu}[(3-2\eta)\eta^2 + 12\Lambda\eta], \quad \psi_4 = \frac{-l}{\mu}[\eta^2(1-\eta) + 6\Lambda\eta(1-\eta)],$$

Bending Rotation Shape Functions:

$$\varphi_1 = \frac{6}{l\mu}(\eta - 1)\eta, \quad \varphi_2 = \frac{1}{\mu}(\mu - 4\eta + 3\eta^2 - 12\Lambda\eta),$$
$$\varphi_3 = \frac{6}{l\mu}(1-\eta)\eta, \quad \varphi_4 = \frac{1}{\mu}(3\eta^2 - 2\eta + 12\Lambda\eta),$$

where $\Lambda = \frac{E_e I_e}{\kappa G_e A_e l^2}$, $\mu = 1 + 12\Lambda$ and $\eta = \frac{x}{l}$. The subscript e designates the value of the parameter in the homogeneous and isotropic