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Upper bound on the radii of black-hole photonspheres

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ABSTRACT

One of the most remarkable predictions of the general theory of relativity is the existence of black-hole "photonspheres", compact null hypersurfaces on which massless particles can orbit the central black hole. We prove that every spherically-symmetric asymptotically flat black-hole spacetime is characterized by a photonsphere whose radius is bounded from above by $r_\gamma \leqslant 3M$, where M is the total ADM mass of the black-hole spacetime. It is shown that hairy black-hole configurations conform to this upper bound. In particular, the null circular geodesic of the (bald) Schwarzschild black-hole spacetime saturates the bound.

1. Introduction

The geodesic motions of test particles around central compact objects (e.g. black holes, neutron stars) provide valuable information about the structure and geometry of the corresponding curved spacetime [1–4]. Of particular importance are circular null geodesics – orbits with constant coordinate radii on which massless particles can orbit the central compact object [5]. Circular null orbits (also known as "photon orbits" or "photonspheres") are interesting from both an astrophysical and theoretical points of view [1–4,6]. For example, the optical appearance to external observers of a compact star undergoing gravitational collapse is closely related to the physical properties of the photonsphere [4,7,8].

Unstable circular null geodesics are also related to the characteristic (quasinormal) resonances of black-hole spacetimes [4,9–14]. In particular, in the framework of the geometric–optics (eikonal) approximation, these characteristic resonances can be interpreted in terms of massless particles temporarily trapped at the unstable null orbit of the black-hole spacetime [4,9–14]. In addition, it was shown [15] that the remarkable phenomenon of strong gravitational lensing by black holes is closely related to the presence of null circular geodesics in the corresponding black-hole spacetimes.

Furthermore, for hairy black-hole spacetimes a theorem was recently proved [16,17] that reveals the central role played by the black-hole photonsphere in determining the effective length of the hair outside the black-hole horizon. According to this theorem, the non-trivial (non-asymptotic) behavior of the hair must extend above the black-hole photonsphere. In this respect, the black-hole photonsphere provides a generic lower bound on the effective length of the black-hole hair [16,17]. In addition, it was recently proved [18] that the innermost null circular geodesic of a black-hole spacetime provides the fastest way (as measured by asymptotic observers) to circle the central black hole.

The central role played by photonspheres in both astrophysical [4,7,8] and cosmological [15] scenarios involving black holes, as well as in purely theoretical studies of black-hole spacetimes [4,9–14,16,18], makes it highly important to explore the physical properties of these unique null orbits. The main goal of the present study is to extend our knowledge about the physical properties of these fascinating geodesics. In particular, in the present Letter we shall address the following question: In a generic black-hole spacetime, how close is the photonsphere to the black-hole horizon? As we shall show below, one can derive a generic *upper bound* on the radii of black-hole photonspheres. This bound is expressed in terms of the total (ADM) mass of the black-hole spacetime.

2. Description of the system

We consider static, spherically symmetric, asymptotically flat black-hole spacetimes. The line element may take the following form in Schwarzschild coordinates [17–19]

$$ds^2 = -e^{-2\delta}\mu\, dt^2 + \mu^{-1}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{1}$$

where the metric functions $\delta(r)$ and $\mu(r)$ depend only on the Schwarzschild areal coordinate r. Asymptotic flatness requires

$$\mu(r\to\infty)\to 1 \quad \text{and} \quad \delta(r\to\infty)\to 0. \tag{2}$$

It is worth emphasizing that our results would be valid for all spherically symmetric asymptotically flat black-hole spacetimes.

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We note, in particular, that we do *not* assume $\delta(r)=0$ (a property which characterizes the familiar Schwarzschild and Reissner–Nordström black-hole spacetimes) and thus our results would be applicable to hairy black-hole configurations as well [in these spacetimes $\delta(r)\neq 0$, see [20–24] and references therein].

Taking $T^t_t=-\rho$, $T^r_r=p$, and $T^\theta_\theta=T^\phi_\phi=p_T$, where ρ, p, and p_T are identified as the energy density, radial pressure, and tangential pressure respectively [25], the Einstein equations $G^\mu_\nu=8\pi T^\mu_\nu$ read

$$\mu'=-8\pi r\rho+(1-\mu)/r, \tag{3}$$

and

$$\delta'=-4\pi r(\rho+p)/\mu, \tag{4}$$

where the prime stands for differentiation with respect to r. (We use natural units in which $G=c=1$.)

Regularity of the event horizon at $r=r_{\rm H}$ requires [17]

$$\mu(r_{\rm H})=0 \quad \text{with } \mu'(r_{\rm H})\geqslant 0, \tag{5}$$

and

$$\delta(r_{\rm H})<\infty; \qquad \delta'(r_{\rm H})<\infty. \tag{6}$$

From Eqs. (3) and (5) one finds the boundary condition [17,26]

$$8\pi r_{\rm H}^2\rho(r_{\rm H})\leqslant 1. \tag{7}$$

The equality sign corresponds to extremal black holes. From Eqs. (4)–(6) one finds the additional boundary condition

$$-p(r_{\rm H})=\rho(r_{\rm H}). \tag{8}$$

The mass $m(r)$ contained within a sphere of radius r is given by

$$m(r)=\frac{1}{2}r_{\rm H}+\int_{r_{\rm H}}^{r}4\pi r'^2\rho(r')\,dr', \tag{9}$$

where $m(r_{\rm H})=r_{\rm H}/2$ is the horizon mass. From Eqs. (3) and (9) one finds the relation

$$\mu(r)\equiv 1-2m(r)/r. \tag{10}$$

A finite mass configuration is characterized by a density profile $\rho(r)$ which approaches zero faster than r^{-3},

$$r^3\rho(r)\to 0 \quad \text{as } r\to\infty. \tag{11}$$

3. Upper bound on the radii of black-hole photonspheres

We shall now consider the following question: In the generic spherically-symmetric black-hole spacetime (1), how close is the photonsphere to the black-hole horizon? To answer this question in the most general form, we shall now prove the existence of a generic *upper bound* on the radius of the innermost null circular geodesic.

We shall first follow the analysis of [2,4,16,18] in order to calculate the location $r=r_\gamma$ of the null circular geodesic for a black-hole spacetime described by the line element (1). The Lagrangian describing the geodesics in the spacetime (1) is given by

$$2\mathcal{L}=-e^{-2\delta}\mu\dot t^2+\mu^{-1}\dot r^2+r^2\dot\phi^2, \tag{12}$$

where a dot stands for differentiation with respect to proper time.

The generalized momenta derived from the Lagrangian (12) are given by [2,4]

$$p_t=-e^{-2\delta}\mu\dot t\equiv -E=\text{const}, \tag{13}$$

$$p_\phi=r^2\dot\phi\equiv L=\text{const}, \tag{14}$$

and

$$p_r=\mu^{-1}\dot r. \tag{15}$$

The Lagrangian is independent of both t and ϕ. This implies that E and L are constants of the motion. The Hamiltonian of the system is given by [2,4] $\mathcal{H}=p_t\dot t+p_r\dot r+p_\phi\dot\phi-\mathcal{L}$, which implies

$$2\mathcal{H}=-E\dot t+L\dot\phi+\mu^{-1}\dot r^2=\epsilon=\text{const}, \tag{16}$$

where $\epsilon=0$ for null geodesics and $\epsilon=1$ for timelike geodesics. Substituting Eqs. (13)–(14) into (16), one finds

$$\dot r^2=\mu\left(\frac{E^2}{e^{-2\delta}\mu}-\frac{L^2}{r^2}\right) \tag{17}$$

for null geodesics.

Circular geodesics are characterized by $\dot r^2=(\dot r^2)'=0$ [2,4]. This yields the relation

$$2e^{-2\delta}\mu-r\left(e^{-2\delta}\mu\right)'=0 \tag{18}$$

for the null circular geodesic. Substituting the Einstein equations (3)–(4) into Eq. (18), one finds the characteristic relation

$$\mathcal{N}(r=r_\gamma)=0 \tag{19}$$

for null circular orbits, where

$$\mathcal{N}(r)\equiv 3\mu-1-8\pi r^2p. \tag{20}$$

We shall henceforth consider the innermost circular null geodesic of the black-hole spacetime. Of course, this geodesic must satisfy the characteristic equation (19). We shall first prove that such circular null geodesic must exist in the black-hole spacetime: taking cognizance of Eqs. (5), (7), (8), and (20), one finds

$$\mathcal{N}(r_{\rm H})\leqslant 0 \tag{21}$$

at the black-hole horizon [27]. In addition, from Eqs. (2), (11), and (20) [see also Eq. (28) below] one finds the asymptotic behavior

$$\mathcal{N}(r\to\infty)\to 2. \tag{22}$$

Inspection of Eqs. (21) and (22) reveals that there must be at least one intermediate radius r_γ (located between the black-hole horizon and spatial infinity) for which $\mathcal{N}(r=r_\gamma)=0$. This radius corresponds to the location of the null circular geodesic, see Eq. (19).

It is worth emphasizing that the null circular geodesic (19) is the limiting case of timelike circular geodesics. That is, the null circular geodesic is the innermost circular orbit in the black-hole spacetime [2,4]. The spacetime region between the black-hole horizon and the photonsphere, $r_{\rm H}\leqslant r<r_\gamma$, in which circular geodesics are excluded, is characterized by the inequality

$$\mathcal{N}(r_{\rm H}\leqslant r<r_\gamma)<0. \tag{23}$$

We shall now derive a generic upper bound on the radii of the innermost null circular geodesics. The conservation equation $T^\mu_{\nu;\mu}=0$ for the energy–momentum tensor has only one non-trivial component [17]

$$T^\mu_{r;\mu}=0. \tag{24}$$