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Three-particle integrable systems with elliptic dependence on momenta and theta function identities

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ABSTRACT

We claim that some non-trivial theta-function identities at higher genus can stand behind the Poisson commutativity of the Hamiltonians of elliptic integrable systems, which were introduced in [1,2] and are made from the theta-functions on Jacobians of the Seiberg–Witten curves. For the case of three-particle systems the genus-2 identities are found and presented in the Letter. The connection with the Macdonald identities is established. The genus-2 theta-function identities provide the direct way to construct the Poisson structure in terms of the coordinates on the Jacobian of the spectral curve and the elements of its period matrix. The Lax representations for the two-particle systems are also obtained.

1. Introduction

We suggest a new approach to obtain the multi-particle generalization of p–q dual integrable systems constructed in [1,3]. The list of these systems includes duals to elliptic Calogero [4] and elliptic Ruijsenaars models [5,6]. The most interesting are the double-elliptic integrable systems [1,2,7,8], where both coordinates and momenta take values on elliptic curves. From the point of view of the low-energy effective actions of Yang–Mills theories [9,10] the integrable systems under consideration are associated with (compactified) six-dimensional SUSY gauge theories [11–19] and are strongly involved in modern discussion of the Seiberg–Witten theory [20–28].

The main claim of our Letter is the existence of some new theta-function identities which stand behind the Poisson commutativity of the Hamiltonians considered in [1,2]. We present such relations for the case of three-particle systems. Unfortunately, it is not clear, how to prove them even in this situation, but these relations seem to be rather interesting by themselves – they can be considered as a genus two generalization of the Macdonald identities [29]. Also, these theta-function identities provide a direct way to construct the Poisson structure in terms of the coordinates on the Jacobian of the spectral curve and the elements of its period matrix. In the case of three-particle systems we reduce the problem of constructing the Poisson brackets to the problem of solving the system of three partial differential equations with respect to the pair of unknown functions.

In Section 2 we discuss the Hamiltonians introduced in [1,2]. The whole construction is restricted to the Seiberg–Witten families of the spectral curves and the Poisson commutativity of the Hamiltonians is related to the Seiberg–Witten prepotential. In [2] this commutativity was checked in the first non-trivial example in the first several orders of Λ/a-expansion – and this check was relied severely on the known shape of the Seiberg–Witten prepotential. In Section 3 we propose another approach, that deals with arbitrary Riemann surfaces instead of the Seiberg–Witten curves. Then the Poisson commutativity of the Hamiltonians is due to some new theta-function identities, which seem to be true for an arbitrary Riemann surface. In Sections 4 and 5 we present the theta-function identities of genus two for the case of three-particle systems and reveal some interesting mathematical structures behind them. In Section 6 the connection with the Ruijsenaars models is established by means of some specific trigonometric limits. In Section 7 we go back to the case of two-particle systems and construct the Lax representation using the original Ruijsenaars' idea [6].

2. Hamiltonians and Seiberg–Witten data

The Hamiltonians of the systems dual to the elliptic Calogero and Ruijsenaars models with an elliptic dependence on momenta as well as the self-dual double elliptic systems were introduced in [1,2]. In the case of $N=3$ particles in the center of a mass frame one has two Hamiltonians of the form

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$$H_1(\mathbf{z}|\Omega) = \frac{\theta^{(2)}_{11}(\mathbf{z}|\Omega)}{\theta^{(2)}(\mathbf{z}|\Omega)}, \qquad H_2(\mathbf{z}|\Omega) = \frac{\theta^{(2)}_{22}(\mathbf{z}|\Omega)}{\theta^{(2)}(\mathbf{z}|\Omega)}, \tag{2.1}$$

where we use the following notations for the Riemann theta function with symmetric 2×2 period matrix Ω:

$$\theta^{(2)}(\mathbf{z}|\Omega) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \mathrm{e}\left(\frac{1}{2}\mathbf{n}^t\Omega\mathbf{n} + \mathbf{n}\cdot\mathbf{z}\right),$$
$$\mathbf{z}^t = (z_1, z_2), \qquad \mathrm{e}(x) \equiv \exp(2\pi\imath x),$$
$$\theta^{(2)}_{\mathbf{c}} \equiv \theta^{(2)}\begin{bmatrix}\mathbf{0}\\ \mathbf{c}/3\end{bmatrix} = \sum_{\mathbf{n}\in\mathbb{Z}^2} \mathrm{e}\left(\frac{1}{2}\mathbf{n}^t\Omega\mathbf{n} + \mathbf{n}\cdot\left(\mathbf{z}+\frac{\mathbf{c}}{3}\right)\right), \quad \mathbf{c}\in\mathbb{Z}^2_3.$$

The role of the non-commutative coordinates and momenta in these Hamiltonians is played by the elements of the period matrix Ω and the coordinates on the Jacobian $\mathbf{z}$. The hypothesis of [1] is that the Hamiltonians are Poisson commuting with respect to the Seiberg–Witten symplectic structure

$$\omega^{\mathrm{SW}} = \sum_{i=1}^{N} \mathrm{d}\hat{p}_i \wedge \mathrm{d}a_i$$

with $\sum_{i=1}^N a_i = 0$ and $z_i = \hat{p}_i - \frac{1}{N}\sum_{j=1}^N \hat{p}_j$, while the $N(N-1)/2 = 3$ elements of the period matrix Ω are not arbitrary, but functions of just $N-1=2$ flat moduli:

$$\Omega = \Omega(\mathbf{a}). \tag{2.2}$$

Following this idea one can assume the Poisson brackets in terms of the coordinates (z_1, z_2) and the elements Ω_{ij} to be of the form

$$\{z_i, \Omega_{jk}\} = P_{ijk}, \quad \{z_i, z_j\} = 0, \quad \{\Omega_{ij}, \Omega_{kl}\} = 0. \tag{2.3}$$

Notice that the elements P_{ijk} are again functions of the flat moduli only. Thus, we can consider them to be the functions of the period matrix Ω as well:

$$P_{ijk} = P_{ijk}(\Omega). \tag{2.4}$$

Taking into account the connection of the period matrix Ω with the Seiberg–Witten prepotential

$$\Omega_{IJ} = \frac{\partial^2\mathcal{F}}{\partial a_I \partial a_J},$$

and the assumption [1] that the prepotential is Poisson-commuting with the total momentum of the system, we get the following relation:

$$P_{IJK} = \frac{\partial^3\mathcal{F}}{\partial a_I \partial a_J \partial a_K}. \tag{2.5}$$

In the Seiberg–Witten theory these third derivatives are usually given by the residue formulas and can be expressed through theta-constants, see, for example, [30].

3. Poisson brackets

In this section we forget about the Seiberg–Witten prepotential and investigate, what can be achieved with the help of (2.3) and (2.4) only. So, we consider Ω as an arbitrary period matrix of genus-2 Riemann surface.

The elements of the Poisson structure (2.3) P_{ijk} are unknown. To derive them as functions of Ω, we use two necessary conditions:

1. The Poisson commutativity of Hamiltonians (2.1),
2. The Jacobi identity.

To impose the first condition we formulate the Poisson bracket between the Hamiltonians (2.1) in terms of the coordinates $\mathbf{z}$ and the elements of the period matrix:

$$\{H_1, H_2\} = \sum_{i=1}^{2}\sum_{j\leqslant k} P_{ijk}\left(\frac{\partial H_1}{\partial z_i}\frac{\partial H_2}{\partial\Omega_{jk}} - \frac{\partial H_2}{\partial z_i}\frac{\partial H_1}{\partial\Omega_{jk}}\right).$$

Then the strategy of finding P_{ijk} is the following. The Poisson commutativity of the Hamiltonians $\{H_1, H_2\} = 0$ means that there are some relations on genus two theta functions with coefficients depending only on the elements of the period matrix. Hence, to satisfy the first condition we search for the following relations on genus two theta functions:

$$\sum_{i=1}^{2}\sum_{j\leqslant k} P^{\alpha}_{ijk}\left(\frac{\partial H_1}{\partial z_i}\frac{\partial H_2}{\partial\Omega_{jk}} - \frac{\partial H_2}{\partial z_i}\frac{\partial H_1}{\partial\Omega_{jk}}\right) = 0, \quad \forall \mathbf{z}, \Omega, \tag{3.1}$$

where index α enumerates the relations and elements P^{α}_{ijk} are some new functions of Ω which can be used for constructing P_{ijk} in the form

$$\{z_i, \Omega_{jk}\} = P_{ijk} = \sum_{\alpha} F_{\alpha} P^{\alpha}_{ijk}, \quad F_{\alpha} = F_{\alpha}(\Omega). \tag{3.2}$$

As we mentioned earlier, it is natural to assume that the coefficients P^{α}_{ijk} are some theta-constants. Thus, the relations (3.1) are made out of the theta functions, theta-constants and their derivatives. Also, the left-hand side of (3.1) should vanish for arbitrary values of $\mathbf{z}$ and Ω, implying that our relations are actually some theta-function identities. It is natural to wonder, if any identities of such form exist without a direct reference to the Seiberg–Witten theory.

Indeed, we found two relations on genus-2 theta functions (2.1) and genus-2 theta constants P^{α}_{ijk}, $\alpha = 1, 2$, which satisfy all the symmetry conditions arising from the previous section. The relations are linearly independent, but the second relation can be obtained from the first one by permutating the diagonal elements of the matrix Ω and the coordinates (z_1, z_2). Unfortunately, it is not clear for us how to prove these relations in general. But even the first simple analysis shows some interesting mathematical structures behind them. We will present some details on these structures in the next two sections.

Once the relations on genus-2 theta functions with coefficients P^{α}_{ijk} are found, we impose the second condition (the Jacobi identity) in order to find $F_{\alpha}(\Omega)$ (3.2). It takes the form

$$\begin{aligned}&\sum_{m\leqslant n}\left(F_1P^1_{1mn} + F_2P^2_{1mn}\right)\partial_{\Omega_{mn}}\left(F_1P^1_{2ij} + F_2P^2_{2ij}\right)\\ &\quad= \sum_{m\leqslant n}\left(F_1P^1_{2mn} + F_2P^2_{2mn}\right)\partial_{\Omega_{mn}}\left(F_1P^1_{1ij} + F_2P^2_{1ij}\right),\end{aligned} \tag{3.3}$$

where $i \leqslant j$, $i, j \in \{1, 2\}$. Thus, we reduce the problem of constructing the Poisson brackets to the problem of solving the system of three partial differential equations with respect to the unknown functions F_{α}, $\alpha = 1, 2$. Of course, it is a challenging issue to find the set of all solutions of the system (3.3) and to understand whether the systems under consideration are unique or not. We leave this problem for future investigation.

4. Theta-function identities of genus 2

In this section we discuss the identities (3.1) on the genus two Riemann theta functions and present the elements P^{α}_{ijk}, $\alpha = 1, 2$, as a solution of the system of linear equations. To describe the