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Finite-volume corrections to charge radii

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ABSTRACT

The finite-volume nature of lattice QCD entails a variety of effects that must be handled in the process of performing chiral extrapolations. Since the pion cloud that surrounds hadrons becomes distorted in a finite volume, hadronic observables must be corrected before one can compare with the experimental values. The electric charge radius of the nucleon is of particular interest when considering the implementation of finite-volume corrections. It is common practice in the literature to transform electric form factors from the lattice into charge radii prior to analysis. However, there is a fundamental difficulty with using these charge radii in a finite-volume extrapolation. The subtleties are a consequence of the absence of a continuous derivative on the lattice. A procedure is outlined for handling such finite-volume corrections, which must be applied directly to the electric form factors themselves rather than to the charge radii.

1. Introduction

Lattice QCD provides important non-perturbative techniques for the analysis of many observables. One of the notable features of lattice QCD is that it must be performed in a finite volume. The associated finite-volume effects can be used to access interesting phenomena. For example, multi-hadron states are only resolvable at finite lattice sizes; the discrete energy eigenvalues become increasingly close together as the box size becomes large. The finite-volume nature of lattice QCD has important consequences, some of which require careful attention. For example, although regularization in both the infrared and ultraviolet regions is an automatic feature of lattice QCD with a finite lattice spacing, finite-sized phenomena, such as the virtual pion clouds that surround hadrons, become distorted. This results in deviations in the values of lattice observables that can become significant in the chiral regime [1–6]. Therefore, a method for correcting finite-volume effects by estimating their size is sought, using a complementary approach, such as chiral effective field theory (χEFT) [3–13].

Study of the quark mass dependence of lattice QCD simulation results can be particularly insightful for examining the chiral properties of hadrons. In relating lattice calculations to experimental results, it is essential to incorporate the low-energy features of QCD in order to obtain reliable extrapolations in both quark mass and volume.

In lattice QCD, form factors are measured at discrete values of momentum transfer, corresponding to the quantization of the momentum modes on the finite spatial volume [10,11,14–22]. Once form factors have been extracted from lattice simulations, they are typically converted directly into charge radii. The essential difficulty lies in the definition of the charge 'radius' at finite volume (more precisely, the slope of the form factor at zero momentum transfer, $Q^2 = \vec{q}^2 - q_0^2$). In order to define the radius, a derivative must be applied to the electric form factor, with respect to a small momentum transfer. This approach breaks down on the lattice, where only discrete momentum values are allowed.

In most cases, calculating the finite-volume corrections poses no essential problems [5,6,13,23–26]. However, because of the absence of a continuous derivative on the lattice, the treatment of the electric charge radius is more subtle [9,27,28]. Therefore, a method is outlined for handling finite-volume corrections to a given lattice simulation result.

Finite-volume charge radii are calculated using the finite-volume electric form factors $G_E^L(Q^2)$, with Q^2 taking an allowed value on the lattice. It will be shown that the finite-volume corrections to the loop integrals must be applied before the conversion from the form factor to the charge radius. An extrapolation in Q^2 is then chosen in order to construct an infinite-volume charge radius, defined in the usual manner.

2. Effective field theory

In heavy-baryon chiral perturbation theory (χPT), it is usual to define the Sachs electromagnetic form factors $G_{E,M}$, which parametrize the matrix element for the quark current J_μ, as

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$$\langle B(p')|J_\mu|B(p)\rangle = \bar{u}^{s'}(p')\left\{\mathrm{v}_\mu G_E(Q^2) + \frac{i\epsilon_{\mu\nu\rho\sigma}\mathrm{v}^\rho S_\mathrm{v}^\sigma q^\nu}{m_B} G_M(Q^2)\right\}u^s(p), \tag{1}$$

where v is the velocity of the baryon and Q^2 is the positive momentum transfer $Q^2 = -q^2 = -(p'-p)^2$. Lattice QCD results are often constructed from an alternative representation, using the form factors F_1 and F_2, which are called the Dirac and Pauli form factors, respectively. The Sachs form factors are simply linear combinations of F_1 and F_2

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4m_B^2}F_2(Q^2), \tag{2}$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2). \tag{3}$$

In the heavy-baryon formulation of Eq. (1), the spin operator, $S_\mathrm{v}^\mu = -\frac{1}{4}\gamma_5[\gamma^\mu,\gamma^\nu]\mathrm{v}_\nu$, is used [29,30]. The momentum transfer dependence in the electric form factor $G_E(Q^2)$, allows a charge radius to be defined in the usual manner

$$\langle r^2\rangle_E = \lim_{Q^2\to 0} -6\frac{\partial G_E(Q^2)}{\partial Q^2}. \tag{4}$$

2.1. Loop integral definitions

The loop integrals, in the continuum limit, that contribute to the electric form factor of the nucleon are invariant under arbitrary translations of the internal momentum $\vec{k}\to\vec{k}+\delta\vec{k}$. However, a finite-volume sum over discrete loop momenta is only invariant if $\delta\vec{k}$ is an allowed value of momentum on the lattice. The loop integrals in the heavy-baryon approximation that correspond to the leading-order diagrams in Figs. 1–3 are obtained by performing the pole integration for k_0

$$\mathcal{T}_N(\vec{q}^{\,2}) = -\frac{\chi_N}{5\pi}\int \mathrm{d}^3k\,\frac{(k^2-\vec{k}\cdot\vec{q})}{\omega_{\vec{k}}\,\omega_{\vec{k}-\vec{q}}(\omega_{\vec{k}}+\omega_{\vec{k}-\vec{q}})}, \tag{5}$$

$$\mathcal{T}_\Delta(\vec{q}^{\,2}) = -\frac{\chi_\Delta}{5\pi}\int \mathrm{d}^3k\,\frac{(k^2-\vec{k}\cdot\vec{q})}{(\omega_{\vec{k}}+\Delta)(\omega_{\vec{k}-\vec{q}}+\Delta)(\omega_{\vec{k}}+\omega_{\vec{k}-\vec{q}})}, \tag{6}$$

$$\mathcal{T}_{\mathrm{tad}}(\vec{q}^{\,2}) = -\frac{\chi_t}{\pi}\int \mathrm{d}^3k\,\frac{1}{\omega_{\vec{k}}+\omega_{\vec{k}-\vec{q}}}, \tag{7}$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2+m_\pi^2}$, m_π is the pion mass, and Δ is the Delta-nucleon mass-splitting. Note that each integral does not explicitly depend on Q^2, but depends on the three-momentum transfer squared, $\vec{q}^{\,2}$. The chiral coefficients χ_N, χ_Δ and χ_t are derived from couplings arising in the Lagrangian of chiral perturbation theory [31]

$$\chi_N^{\mathrm{prot}} = -\frac{5}{16\pi^2 f_\pi^2}(D+F)^2 = -\chi_N^{\mathrm{neut}}, \tag{8}$$

$$\chi_\Delta^{\mathrm{prot}} = +\frac{5}{16\pi^2 f_\pi^2}\frac{4\mathcal{C}^2}{9} = -\chi_\Delta^{\mathrm{neut}}, \tag{9}$$

$$\chi_t^{\mathrm{prot}} = -\frac{1}{16\pi^2 f_\pi^2} = -\chi_t^{\mathrm{neut}}, \tag{10}$$

where the value of the pion decay constant is $f_\pi = 92.4$ MeV. The values for the couplings are estimated from the SU(6) flavor-symmetry relations [30,32] and from phenomenology: $D = 0.76$, $F = \frac{2}{3}D$ and $\mathcal{C} = -2D$.

To obtain the integrals that allow the determination of the quark mass expansion of the electric charge radius, one takes the derivative of each infinite-volume integral $\mathcal{T}(\vec{q}^{\,2})$, with respect to momentum transfer $\vec{q}^{\,2}$, as $\vec{q}^{\,2}\to 0$:

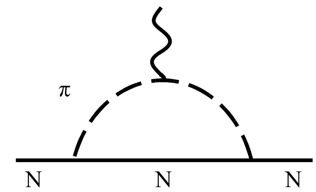

Fig. 1. The pion loop contributions to the electric charge radius of a nucleon. All charge conserving pion–nucleon transitions are implicit.

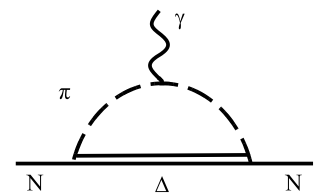

Fig. 2. The pion loop contribution to the electric charge radius of a nucleon, allowing transitions to the nearby and strongly-coupled Δ baryons.

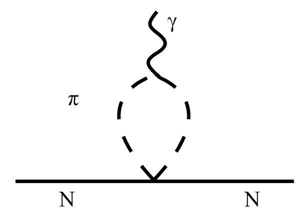

Fig. 3. The tadpole contribution at $\mathcal{O}(m_q)$ to the electric charge radius of a nucleon.

$$T = \lim_{\vec{q}^{\,2}\to 0} -6\frac{\partial\mathcal{T}(\vec{q}^{\,2})}{\partial\vec{q}^{\,2}}. \tag{11}$$

This derivative is equivalent to that of Eq. (4) in the Breit frame, defined by zero energy transfer to the nucleon ($q^\mu = (0,\vec{q})$). Using the derivative forms, T, therefore allows one to recover the familiar chiral expressions for the quark mass dependence of the charge radii.

2.2. Finite-volume corrections

In the analysis of finite-volume effects within χEFT, one requires an evaluation of the correction associated with replacing the continuum loop integrals by finite sums. This correction is expressed in the form

$$\delta_L[\mathcal{T}] = \chi\left[\frac{(2\pi)^3}{L_xL_yL_z}\sum_{k_x,k_y,k_z} - \int \mathrm{d}^3k\right]\mathcal{I}, \tag{12}$$

for an integrand, $\mathcal{I}$. This is not so straightforward in the case of the charge radius, which involves a $\vec{q}^{\,2}$ derivative. Because of the fact that only certain, discrete values of momenta are allowed on the lattice, only a finite-difference equation may be constructed from these allowed momenta. The finite-difference equation, ideally, would be constructed from the lowest value of $\vec{q}^{\,2}$ available on the lattice, $\vec{k}^2_{\mathrm{min}} = (2\pi\vec{n}/L)^2$, where $\vec{n}$ is a unit lattice vector. This is not possible to do in the Breit frame, where the lowest $\vec{q}^{\,2}$ has a magnitude of $2(2\pi/L)^2$, which, on a moderate lattice size of 3 fm, is approximately $(0.58\ \mathrm{GeV})^2$. In order to obtain a suitable estimate of the slope of the form factor at $\vec{q}^{\,2} = 0$, a procedure is outlined for evaluating finite-volume corrections using the lowest available $\vec{q}^{\,2}$ value. Since the finite-volume corrections are applied directly to