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Estimates of the Vigdergauz constants for regular triangular honeycombs

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ABSTRACT

In this paper we study the effective elastic properties of regular triangular honeycombs. In particular we obtain some simple approximate formulae for the corresponding Vigdergauz constants with accuracy better than 1% for all densities.

1. Introduction and main results

Consider a two-dimensional hexagonal symmetric and periodic structure formed by an isotropic solid material with plane strain bulk modulus K and shear modulus G. It is well-known that the corresponding effective elastic properties of such a structure are equal to that of a transversely isotropic homogeneous material with in-plane elastic moduli (K^*,G^*). In [11], Vigdergauz showed the following general relations between (K,G) and (K^*,G^*):

$$\frac{1}{K^*}=\frac{1}{K}+A_1\left(\frac{1}{K}+\frac{1}{G}\right), \tag{1}$$

$$\frac{1}{G^*}=\frac{1}{G}+A_2\left(\frac{1}{K}+\frac{1}{G}\right), \tag{2}$$

where A_1 and A_2 are positive constants which depend only on the geometry. These constants will be referred to as the Vigdergauz constants.

In this paper we discuss a special type of such structures, namely regular triangular honeycombs honeycomb structure, which are obtained by replicating equilateral triangular cells (see Fig. 1a). For low density triangular honeycombs there exist some simplified formulae, which can be rewritten in the following form (c.f. [4,10]):

$$\frac{1}{K^*}=\frac{1}{2\sqrt{3}}\left(\frac{l}{b}\right)\left(\frac{1}{K}+\frac{1}{G}\right), \tag{3}$$

$$\frac{1}{G^*}=\frac{1}{\sqrt{3}}\left(\frac{l}{b}\right)\left(\frac{1}{K}+\frac{1}{G}\right). \tag{4}$$

Here, l is the height of one period of the structure and b is the thickness of the cell walls (see Fig. 1b). Even though these formulae are relatively correct when $b/l \ll 1$, they are clearly not on the form of (1) and (2). This lack of consistency may be due to the fact that before 1999 nobody knew that the effective elastic moduli (K^*,G^*) and the local moduli (K,G) were linked to each other by the strikingly simple relations (1) and (2). Without this information it was probably tempting to ignore effects caused by the fact that the local Poisson's ratio may be non-zero. In this paper we adjust the simplified models behind these formulae and obtain new formulae of the same simplicity, which fortunately happen to be of the form (1) and (2). In fact, we find the following approximations of A_1 and A_2, respectively:

$$A_{f,1}=\frac{1}{2\sqrt{3}}\frac{l}{b}-\frac{1}{2},$$

$$A_{f,2}=\frac{1}{\sqrt{3}}\frac{l}{b}-\frac{1}{2}.$$

It turns out that $A_{f,i} > A_i > A_{i,HS^+}$, where A_{i,HS^+} is the Vigdergauz constant corresponding to the upper Hashin–Shtrikman bounds for K^* (if $i = 1$) or G^* (if $i = 2$). By observing that the relative deviation between $A_{f,i}$ and A_{i,HS^+} vanishes as $b/l \to 0$, we are able to prove that the triangular honeycomb becomes stiffest possible (among transversely isotropic structures with the same density) as the density decreases.

The above formulae turn out to be particularly useful for the purpose of developing accurate formulae for the Vigdergauz constants for arbitrary densities. By comparing $A_{f,i}$ with numerically estimated values of A_i, the following improved formulae are found:

$$A_{f,1}^{\mathrm{imp}}=\begin{cases} A_{f,1}\left[1-0.3249\frac{b}{l}+1.156\left(\frac{b}{l}\right)^2-5.121\left(\frac{b}{l}\right)^3\right] & \text{for } 0<p_c<0.9,\\ 5.211\left(\frac{b}{l}-\frac{1}{\sqrt{3}}\right)\left(\frac{b}{l}-0.5737\right) & \text{for } 0.9\leqslant p_c\leqslant 1,\end{cases}$$

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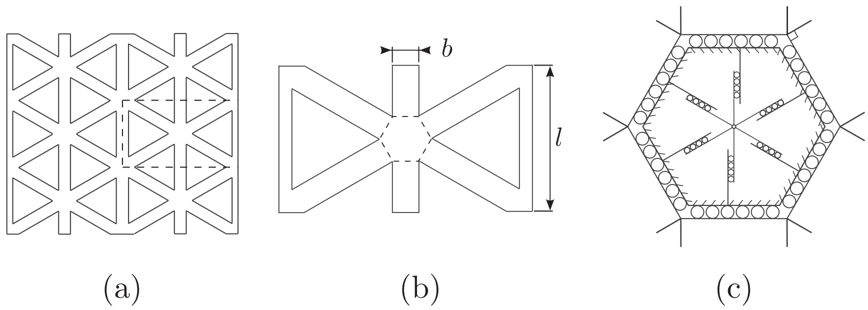

Fig. 1. A regular triangular honeycomb with cell-wall length l and thickness b (a), the unit cell with the hexagonal joint area marked with dashed lines (b), and the simplified joint area (c).

$$A_{f,2}^{\mathrm{imp}} = \begin{cases} A_{f,2}\left[1-0.3964\frac{b}{l}-5.372\left(\frac{b}{l}\right)^2+4.664\left(\frac{b}{l}\right)^3\right] & \text{for } 0<p_c<0.9, \\ 8.854\left(\frac{b}{l}-\frac{1}{\sqrt{3}}\right)\left(\frac{b}{l}-0.5678\right) & \text{for } 0.9\leqslant p_c\leqslant 1, \end{cases}$$

where p_c is the volume fraction of the connected material. The volume fraction as a function of the variable b/l and its inverse are

$$p_c = 1-\left(1-\sqrt{3}\left(\frac{b}{l}\right)\right)^2, \quad \frac{b}{l} = \frac{1}{\sqrt{3}}(1-\sqrt{1-p_c}).$$

The improved formula $A_{f,i}^{\mathrm{imp}}$, which is expressed by a Laurent polynomial in the variable b/l, turns out to be relatively close to the true values of A_i for all densities. In fact, the relative error is in principle less than 1%.

The paper is organized as follows. In Section 2, the model problem is presented. The relevant results from the homogenization theory applied to elasticity are also included. In Section 3, the low-density estimates of the Vigdergauz constants are derived. In Section 4, these estimates are compared with the optimal values given in the Hashin–Shtrikman bounds. In Section 5, the estimates $A_{f,i}$ are compared with numerically estimated values of A_i. The numerical data are then used in Section 6 to find improved versions of the formulae for $A_{f,i}$.

2. The model problem

The honeycomb is assumed to be locally isotropic and linearly elastic. In the plane theory of elasticity such a material's constitutive equation of the stress $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})$ and the strain $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})$ can be written on matrix form as

$$\sigma = \begin{pmatrix} K+G & K-G & 0 \\ K-G & K+G & 0 \\ 0 & 0 & G \end{pmatrix}\varepsilon, \tag{5}$$

where $K, G \in \mathbf{R}_+$ are the bulk and shear modulus in plane strain, respectively. We have

$$K = \frac{E}{2(1+\nu)(1-2\nu)},$$

and

$$G = \frac{E}{2(1+\nu)},$$

where E is the Young's modulus and ν is the Poisson's ratio. The local bulk modulus K is related to the bulk modulus k in three dimensions by $K = k + G/3$. The third component of the strain vector γ_{12} equals $2\varepsilon_{12}$, where ε_{12} is the shear strain. Expositions of the linear theory of elasticity can be found in [6,7].

For one period of a periodic structure, see Fig. 1b, one can consider the problem of finding the displacement which gives a prescribed average strain. The solution of such a problem is known to exist and it gives in this case an isotropic Cartesian tensor relating the average stress to the average strain. The tensor is called the effective tensor. The relation can be written as follows with the same matrix representation as was used above. Let the average stress and the average strain be written $\langle\sigma\rangle = (\langle\sigma_{11}\rangle, \langle\sigma_{22}\rangle, \langle\sigma_{12}\rangle)$ and $\langle\varepsilon\rangle = (\langle\varepsilon_{11}\rangle, \langle\varepsilon_{22}\rangle, \langle\gamma_{12}\rangle)$, respectively. We have the effective constitutive relation

$$\langle\sigma\rangle = \begin{pmatrix} K^*+G^* & K^*-G^* & 0 \\ K^*-G^* & K^*+G^* & 0 \\ 0 & 0 & G^* \end{pmatrix}\langle\varepsilon\rangle, \tag{6}$$

where $K^*, G^* \in \mathbf{R}_+$ are called the effective bulk and shear modulus, respectively. These are the macroscopic elastic properties of the structure. For the facts mentioned above, we refer to [2,5,8].

Suppose that the structure is deformed in a way that makes the strain vector periodic and such that $\langle\varepsilon_{11}\rangle = \langle\varepsilon_{22}\rangle$. Then Eq. (6) implies that

$$K^* = \frac{\langle\sigma_{22}\rangle}{2\langle\varepsilon_{22}\rangle}. \tag{7}$$

On the other hand, if the deformation is such that the strain vector is periodic and satisfies $\langle\varepsilon_{11}\rangle = -\langle\varepsilon_{22}\rangle$, Eq. (6) implies that

$$G^* = \frac{\langle\sigma_{22}\rangle}{2\langle\varepsilon_{22}\rangle}. \tag{8}$$

Under the assumption of Eq. (6), the conditions that the average strain should satisfy $\langle\varepsilon_{11}\rangle = \langle\varepsilon_{22}\rangle$ or $\langle\varepsilon_{11}\rangle = -\langle\varepsilon_{22}\rangle$ are equivalent to restricting the average stress to satisfy $\langle\sigma_{11}\rangle = \langle\sigma_{22}\rangle$ or $\langle\sigma_{11}\rangle = -\langle\sigma_{22}\rangle$, respectively.

To simplify the analysis additional assumptions are made. These assumptions will be referred to as the simplified model. The hexagonal joint area, marked with dashed lines in Fig. 1b, is assumed to be freely deformable under three restrictions. First, the joint area is assumed to remain hexagonal but not necessarily regular. Secondly, the side lengths of the joint area are assumed to remain equal to the thicknesses of the connecting cell walls. Thirdly, the cell walls are assumed to remain perpendicular to the walls of the joint area. An illustration of the simplified joint area is shown in Fig. 1c.

3. Estimates of the Vigdergauz constants

In this section estimates of the Vigdergauz constants will be presented. The approach that will be used is to apply external loads to the structure in the simplified model. This is done in such a way that estimates of the effective bulk modulus K^* and the effective shear modulus G^* are obtained from the ratio of the average strain and the average stress. Then by using the relations (1) and (2) estimates of the A_1 and A_2 follow.