Physics Letters B 725 (2013) 468–472

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Symmetries and gravitational Chern–Simons Lagrangian terms

L. Bonora [a], M. Cvitan [b], P. Dominis Prester [c,*], S. Pallua [b], I. Smolić [b]

[a] International School for Advanced Studies (SISSA/ISAS), Via Bonomea 265, 34136 Trieste, Italy
[b] Theoretical Physics Division of Particles and Fields, Physics Department, Faculty of Science, University of Zagreb, Bijenička cesta 32, HR-10002 Zagreb, Croatia
[c] Department of Physics, University of Rijeka, Radmile Matejčić 2, HR-51000 Rijeka, Croatia

ARTICLE INFO

Article history:
Received 8 May 2013
Received in revised form 11 July 2013
Accepted 15 July 2013
Available online 19 July 2013
Editor: M. Cvetič

ABSTRACT

We consider some general consequences of adding pure gravitational Chern–Simons term to manifestly diff-covariant theories of gravity, focusing essentially on spacetimes with $D > 3$. Extending the result of a previous paper we enlarge the class of metrics for which the inclusion of a gravitational Chern–Simons term in the action does not affect solutions and corresponding physical quantities. In the case in which such solutions describe black holes (of general horizon topology) we show that the black hole entropy is also unchanged. We arrive at these conclusions by proving three general theorems and studying their consequences. One of the theorems states that the contribution of the gravitational Chern–Simons to the black hole entropy is invariant under local rescaling of the metric.

1. Introduction

This Letter is a follow up of previous papers in which we have analyzed the consequences of adding a purely gravitational Chern–Simons (gCS) term [1] to a manifestly diffeomorphism-invariant gravitational action in $(4k-1)$-dimensional spacetime, which is a generalization of the idea originally introduced in $D = 3$ dimensions in [2,3]. Following a proposal by Tachikawa [4], in [5] we analyzed the general consequences of adding one such gCS term to the action, in particular the appearance of a new contribution to the thermodynamical entropy. In [6] we considered the global geometrical aspects implied by the presence of a gCS term, both at the level of the action and the entropy, and studied the topological conditions for the well-definiteness of both. Except in the three-dimensional case, very well studied in the literature, it does not seem to be easy to see the effects of a gCS term on observables. In the simplest and more symmetric cases they appear to be null. For this reason in [7] we studied the case of Myers–Perry black hole in seven dimensions. We were able, at least perturbatively, to show that in some sufficiently complicated configuration the effects of the gCS are not identically vanishing.

In this Letter we would like to enlarge the null effect results of [8], with the purpose of circumscribing as closely as possible the cases in which the addition of a gCS is irrelevant from an observational point of view. More to the point we are interested in the gravity theories in $D = 2n-1$ dimensions ($n \in 2\mathbb{N}$) with Lagrangians of the form

$$\mathbf{L} = \mathbf{L}_0 + \lambda \mathbf{L}_{\mathrm{gCS}} \tag{1}$$

where $\mathbf{L}_0$ is some general manifestly diffeomorphism-invariant Lagrangian density and $\mathbf{L}_{\mathrm{gCS}}$ is the gCS Lagrangian density. In (1) λ denotes the gCS coupling constant, whose properties are studied in detail in [6] (and in the special case of $D = 3$ in [2,3,9]). gCS terms have a remarkable set of properties, among which the most notable are: they are not manifestly diff-covariant, though they preserve diff-covariance in the bulk; they have a topological nature which leads to a quantization of their coupling; they are parity-odd and so break parity symmetry; they are conformally covariant, in the sense that under a Weyl rescaling of the metric

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{2}$$

Chern–Simons density transforms as (see [1,10]),

$$\mathbf{L}_{\mathrm{gCS}}[\tilde{\mathbf{\Gamma}}] = \mathbf{L}_{\mathrm{gCS}}[\mathbf{\Gamma}] + d(\cdots) \tag{3}$$

It is clear that gCS Lagrangian terms have a peculiar role in the set of all possible higher-curvature gravity terms, which makes them deserve special attention.

In the following we shall explicitly refer mainly to irreducible gCS terms, whose Lagrangian density is given by

$$\mathbf{L}_{\mathrm{gCS}}[\mathbf{\Gamma}] = n \int_0^1 dt\ \mathrm{str}\big(\mathbf{\Gamma} \mathbf{R}_t^{n-1}\big) \tag{4}$$

* Corresponding author.
E-mail addresses: bonora@sissa.it (L. Bonora), mcvitan@phy.hr (M. Cvitan), pprester@phy.uniri.hr (P. Dominis Prester), pallua@phy.hr (S. Pallua), ismolic@phy.hr (I. Smolić).

0370-2693/$ – see front matter
http://dx.doi.org/10.1016/j.physletb.2013.07.036

Here $\mathbf{R}_t = t\,d\boldsymbol{\Gamma} + t^2\boldsymbol{\Gamma}\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}$ is the Levi-Civita connection and str denotes a symmetrized trace, which is an irreducible invariant symmetric polynomial of the Lie algebra of the $SO(1, D-1)$ group, and all products are wedge products. A general gCS term is a linear combination of irreducible and reducible terms, where the form of the latter is obtained from $(D+1)$-dimensional relation

$$d\mathbf{L}_{\mathrm{gCS}} = \mathrm{tr}\big(\mathbf{R}^{m_1}\big)\cdots\mathrm{tr}\big(\mathbf{R}^{m_k}\big), \quad 2\sum_{j=1}^{k} m_j = D+1,\; m_j \in 2\mathbb{N} \tag{5}$$

with $k > 1$ ($k = 1$ gives irreducible gCS term). For example, in $D = 7$ aside from the irreducible there is also a reducible gCS term.[1] We shall state in what way the obtained results extend to reducible gCS terms.

As anticipated above, in this Letter we want to improve on the results found in [8]. Our aim is to identify the class of metrics for which the inclusion of a gCS term in (1) does not affect solutions and corresponding physical quantities. We show that for a large class of solutions, the effect of a gCS Lagrangian term is in fact null, and solutions corresponding to the Lagrangian $\mathbf{L}_0$ are also solutions corresponding to (1). In the case in which such solutions describe black holes (of general horizon topology) we shall show that the black hole entropy is also unchanged. As the case $n = 2$ ($D = 3$) has already been studied in detail in literature (see, e.g., [2,3,11–13]), we focus here on $n \geqslant 4$ ($D \geqslant 7$).[2] A particularly important intermediate result is the remark that the terms representing the gCS contribution to the entropy is invariant under local rescaling of the metric.

The Letter is organized as follows. In Section 2 we state and prove three theorems on the vanishing properties of the generalized Cotton tensor, the Weyl invariance of the gCS entropy and the vanishing of the latter under some general conditions. In Section 3 we apply such theorems to various physical situations and in Section 4 to linearized equations of motion around some backgrounds.

2. Three theorems

The Letter is based on three results that we state in the form of theorems. The first concerns the effects of a gCS term on the generalized Cotton tensor. The second the invariance of the gCS entropy contribution under Weyl rescalings of the metric. Thanks to these result the third states that metrics such as those in the first theorem do not contribute to the gCS entropy.

2.1. Equations of motion

Adding a gCS term in the Lagrangian brings about additional terms in the equations of motion. It was shown in [10] that the equation for the metric tensor $g_{\alpha\beta}$ acquires an additional term $C^{\alpha\beta}$, which, for the irreducible gCS term (4), is of the form

$$C^{\alpha\beta} = -\frac{n}{2^{n-1}}\epsilon^{\mu_1\cdots\mu_{2n-2}(\alpha}\nabla_\rho\big(R^{\beta)}{}_{\sigma_1\mu_1\mu_2}R^{\sigma_1}{}_{\sigma_2\mu_3\mu_4}\cdots \times R^{\sigma_{n-3}}{}_{\sigma_{n-2}\mu_{2n-5}\mu_{2n-4}}R^{\sigma_{n-2}\rho}{}_{\mu_{2n-3}\mu_{2n-2}}\big) \tag{6}$$

Under the Weyl rescaling of metric (2) the tensor $C^{\alpha\beta}$ transforms as

$$C^{\alpha\beta}[\tilde{g}] = \Omega^{-(D+2)}C^{\alpha\beta}[g] \tag{7}$$

Aside from being conformally covariant, the tensor $C^{\alpha\beta}$ is also traceless and covariantly conserved and so may be considered as a generalization of the Cotton tensor to $D = 4k-1$ dimensions for $k > 1$ [10].

It has been shown in the literature that due to their special symmetry properties, gCS contributions to equations of motion (6) vanish for whole classes of metrics, such as maximally symmetric spaces (and conformally connected metrics) [10], and spherically symmetric metrics (with $SO(D-1)$ isometry subgroup) [8,18]. Here we want to show that there is a much broader class of metrics for which tensor $C^{\alpha\beta}$ vanishes. This is guaranteed by the following theorem.

Theorem 1. *Assume that the metric of D-dimensional spacetime $(M, g_{\mu\nu})$ can be cast, in some region $\mathcal{O} \subset M$, in the following form,*

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu = D(x)\big(A(z)g_{ab}(y)\,dy^a\,dy^b + B(y)h_{ij}(z)\,dz^i\,dz^j\big), \tag{8}$$

where local coordinates on $\mathcal{O}$ are split into $x^\mu = (y^a, z^i)$, $\mu \in \{1,\dots,D\}$, $a \in \{1,\dots,p\}$, and $i \in \{1,\dots,q\}$ (so that $p + q = D$). The functions $B(y)$, $g_{ab}(y)$ and $A(z)$, $h_{ij}(z)$ depend only on the $\{y^a\}$ and $\{z^i\}$ coordinates, respectively. If $p \geqslant 2$ and $q \geqslant 2$ then for all $x \in \mathcal{O}$

$$C^{\mu\nu}(x) = 0 \tag{9}$$

Proof. Due to property (7), equality (9) is preserved under Weyl rescalings (2). By taking $\Omega^2 = (DAB)^{-1}$ the metric (8) may be put in the direct product form

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}\,dx^\mu\,dx^\nu = g_{ab}(y)\,dy^a\,dy^b + h_{ij}(z)\,dz^i\,dz^j \tag{10}$$

so we only have to prove that the theorem holds for the metrics of the type (10). This greatly simplifies our job because both Riemann tensor and its covariant derivative are completely block-diagonal, and as a consequence also the tensor

$$\nabla_\rho\big(R^\beta{}_{\sigma_1\mu_1\mu_2}R^{\sigma_1}{}_{\sigma_2\mu_3\mu_4}\cdots R^{\sigma_{n-3}}{}_{\sigma_{n-2}\mu_{2n-5}\mu_{2n-4}}R^{\sigma_{n-2}\rho}{}_{\mu_{2n-3}\mu_{2n-2}}\big) \tag{11}$$

present in the definition of $C^{\mu\nu}$ (6) is. This means that the components of the tensor in (11) are nonvanishing only when all the indices are either from the y-subspace or the z-subspace. Because there are $D-1$ free indices in (11) which are contracted with the totally antisymmetric Levi-Civita tensor in (6) (that is, all have to be mutually different) it is obvious that when both $p > 1$ and $q > 1$ then (11) is zero, implying that $C^{\mu\nu}$ is also zero. □

For reducible gCS terms Theorem 1 gets modified, allowing other possibilities (aside p or q equal to 0 or 1) in which one may have $C^{\alpha\beta} \neq 0$ for geometries of the type (8). Their contribution to $C^{\alpha\beta}$ is a sum of terms which are of the form [8]

$$\epsilon^{\mu_1\cdots\mu_{D-1}(\alpha}\big(\mathrm{tr}\big(\mathbf{R}^{2m_1}\big)\cdots\mathrm{tr}\big(\mathbf{R}^{2m_{k-1}}\big)\big)_{\mu_1\cdots\mu_{D+1-2m_k}} \times \nabla_\rho\big(R^{\beta)}{}_{\sigma_1\mu_{D+2-2m_k}\mu_{D+3-2m_k}}\cdots R^{\sigma_{n-2}\rho}{}_{\mu_{D-2}\mu_{D-1}}\big) \tag{12}$$

Following the same logic as above it is easy to conclude that for the reducible gCS term, defined implicitly by (5), exceptions to Theorem 1 may appear when there is a subset $\{m_{j_1},\dots,m_{j_l}\}$ of the set of exponents $\{m_j,\ j = 1,\dots,k\}$ such that

$$2\sum_{r=1}^{l} m_{j_r} + \sigma = p \text{ or } q \tag{13}$$

[1] In string theories compactified to $D = 7$, they appear in combination when gCS terms are present.

[2] In string theory $n = 2$ gravitational CS terms play an important and unique role in some black hole analyses (see, e.g., [12,14–17]).