Physics Letters B 724 (2013) 310–315

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Hamiltonian formulation of exactly solvable models and their physical vacuum states

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A R T I C L E I N F O

Article history:
Received 28 September 2012
Received in revised form 16 May 2013
Accepted 19 May 2013
Available online 3 June 2013
Editor: B. Grinstein

Keywords:
Solvable models
Hamiltonian formulation
Exact solution
Vacuum structure

A B S T R A C T

We study simple two-dimensional models with massless and massive fermions in the Hamiltonian framework. While our ultimate goal is to gain a deeper insight to structural differences between the usual ("spacelike" – SL) and light-front (LF) forms of the relativistic dynamics, an attempt is also made to clarify a few conceptual problems of quantum field theory. We point out that contrary to the assumption of canonical quantization, interacting Heisenberg fields do not always reduce to free fields at $t = 0$. We also show that by incorporating operator solutions of the field equations to the canonical formalism, SL and LF Hamiltonians of the derivative-coupling model as well as of the Federbush model acquire an equivalent structure. In the usual canonical treatment, physical predictions in the two schemes disagree – the SL Hamiltonians contain interaction terms while their LF counterparts do not. Using a Bogoliubov transformation, the physical vacuum of the Thirring model is then derived for the first time. It has a form of a coherent state quadratic in composite boson operators which, after bosonization of the vector current, are present in the (nondiagonal) interaction Hamiltonian. To find the vacuum of the Federbush model by an analogous Bogoliubov transformation, we propose a massive version of Klaiber's current bosonization and demonstrate advantages of the LF treatment of the model.

1. Introduction

The usual "spacelike" (SL) and the light-front (LF) [1] forms of relativistic quantum field theory (QFT) are two independent representations of the same physical reality. There are however striking differences between both schemes already at the level of basic properties [2,3]. This concerns the mathematical structure as well as some physical aspects (nature of field variables, division of the Poincaré generators into the kinematical and dynamical sets, status of the vacuum state, etc.) Exactly solvable models offer an opportunity to study the structure of the two theoretical frameworks and their relationship since in these models exact operator solutions of field equations are known. From the solutions, the correlation functions can be computed nonperturbatively and independently of more sophisticated conformal QFT methods [4]. Note that not all solvable models belong to the conformal class. Thus investigations of their properties in a Hamiltonian approach is a very useful alternative. It permits us to study directly the role of the vacuum state and of the operator structures in both forms of QFT. Let us recall in this connection that in the LF form of the relativistic dynamics, Fock vacuum is often the lowest-energy eigenstate of the *full* Hamiltonian. This unique feature is not present in the SL theory and the (unknown) true vacuum state is in practice often replaced by the lowest-energy eigenstate of the free Hamiltonian (perturbative vacuum) without a deeper justification.

In the present Letter, we give a brief survey of a study, based on the above ideas, of the derivative-coupling model (DCM) [5], the Thirring (TM) [6] and the Federbush model (FM) [7]. All these models are quantum field theories in one space dimension. The unifying idea is to benefit from the knowledge of operator solutions of the field equations to re-express the corresponding SL and LF Hamiltonians purely in terms of true degrees of freedom, namely the free fields. This previously overlooked aspect not only simplifies the overall physical picture but also removes structural differences between SL and LF Hamiltonians. For example, in the case of the simplest theory, the DC model, the conventional canonical procedure applied to the SL and LF Lagrangians leads to a striking result: the SL Hamiltonian contains an interaction term while its LF analog does not. On the other hand, if we modify this procedure as suggested above, the discrepancy disappears:

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0370-2693/$ – see front matter
http://dx.doi.org/10.1016/j.physletb.2013.05.045

the SL version of the DCM Hamiltonian is found to also have the interaction-free form. Consequently, the physical SL vacuum of this extremely simple model coincides with the Fock vacuum in a full agreement with the LF result. However, for the models with more complicated interaction structure, the Fock vacuum is an eigenstate only of the free part of the SL Hamiltonians. This is because the interaction parts of the SL Hamiltonians are generally nondiagonal when expressed in terms of creation and annihilation operators. To find the true vacuum state, they have to be diagonalized. This is a complicated dynamical problem which however turns out to be tractable analytically for the Thirring and Federbush models. Our idea is to bring their Hamiltonians to a quadratic form by bosonization of the vector current and to diagonalize them by a Bogoliubov transformation, generating thereby the true ground state as a transformed Fock vacuum (a coherent state). We will show this explicitly for the Thirring model. As for the Federbush model, the conventional procedure yields a vanishing interaction Hamiltonian for the LF case and a nonvanishing one for the SL case. Although this discrepancy is removed when the solutions of the field equations are taken into account, leading to interaction Hamiltonians of the same structure, the LF scheme maintains clear computational advantages with its much simpler operator structures and with the Fock vacuum being its physical vacuum state. We will discuss the Federbush model only very briefly in the present Letter leaving a more detailed treatment for a subsequent publication [8].

On a more formal level, the knowledge of the explicit form of the operator solutions in the studied models tells us that the interacting Heisenberg field does not reduce to a free field at $t=0$, contrary to the assumption of canonical quantization. This may have consequences for more complicated models. Finally, the solvability of the (conformally-noninvariant) massive Federbush model allows us to test the methods of conformal field theory where the mass term is treated as a perturbation [4].

2. The derivative-coupling model

It is instructive to explain our main ideas within a very simple theory – massive fermion and scalar fields interacting via a gradient coupling. Its classical Lagrangian and field equations are

$$\mathcal{L} = \overline{\Psi}\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 - g\partial_\mu\phi J^\mu, \tag{1}$$

$$i\gamma^\mu\partial_\mu\Psi = m\Psi + g\partial_\mu\phi\gamma^\mu\Psi, \tag{2}$$

$$\partial_\mu\partial^\mu\phi + \mu^2\phi = g\partial_\mu J^\mu. \tag{3}$$

The original Schroer's model [5] had $\mu=0$. Our convention for the gamma matrices is $\gamma^0=\sigma^1$, $\gamma^1=i\sigma^2$, $\alpha^1=\gamma^5=\gamma^0\gamma^1$ and σ^i are the Pauli matrices. $J^\mu(x)$ is the vector current composed from the interacting fermion fields, $J^\mu(x)=\overline{\Psi}(x)\gamma^\mu\Psi(x)$. Classically, the vector current is conserved, $\partial_\mu J^\mu=0$, and the scalar field satisfies the free Klein–Gordon equation. This feature is not guaranteed to persist on the quantum level. Since Eq. (2) can be solved exactly irrespectively of whether the scalar field $\phi(x)$ is free or interacting, the most natural way of solving the coupled equations (2) and (3) is to use this solution in the correctly defined (regularized) quantum current that will be inserted to the right-hand side of (3). More specifically, the (classical) solution of the Dirac equation (2) is

$$\Psi(x) = e^{-ig\phi(x)}\psi(x), \qquad i\gamma^\mu\partial_\mu\psi(x) = m\psi(x). \tag{4}$$

In quantum theory, the Fock decomposition of the free massive fermion field $\psi(x)$ has the form

$$\psi(x) = \int_{-\infty}^{+\infty} d\tilde{p}^1\left[b(p^1)u(p^1)e^{-i\hat{p}\cdot x} + d^\dagger(p^1)v(p^1)e^{i\hat{p}\cdot x}\right]. \tag{5}$$

It contains the spinors $u^\dagger(p^1)=(\sqrt{p^-},\sqrt{p^+})$, $v^\dagger(p^1)=(-\sqrt{p^-},\sqrt{p^+})$, where $p^\pm = E(p^1)\pm p^1$, $E(p^1)=\sqrt{p_1^2+m^2}$. In the expansion (5), $\hat{p}\cdot x = E(p^1)t - p^1x^1$ and we have used the abbreviation $d\tilde{p}^1 \equiv dp^1/\sqrt{4\pi E(p^1)}$. The fermion and antifermion Fock operators satisfy the anticommutation relations

$$\{b(p^1), b^\dagger(q^1)\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1-q^1). \tag{6}$$

Similarly, the free scalar field, quantized by $[a(k^1), a^\dagger(l^1)] = \delta(k^1-l^1)$, will be expanded as

$$\phi(x) = \int_{-\infty}^{+\infty} d\tilde{k}^1\left[a(k^1)e^{-i\hat{k}\cdot x} + a^\dagger(k^1)e^{i\hat{k}\cdot x}\right] \equiv \phi^{(+)}(x) + \phi^{(-)}(x). \tag{7}$$

The quantum version of the above Lagrangian contains operators whose products are singular if their space–time arguments coincide. A convenient regularization is to separate these arguments by a small amount ϵ (the "point-splitting"). In quantum theory, the solution $\Psi(x)$ (4) has to be regularized, too. A consistent way to do that is to normal-order the exponential in this solution:

$$\Psi(x) = Z^{1/2}(\epsilon)e^{-ig\phi^{(-)}(x)}e^{-ig\phi^{(+)}(x)}\psi(x), \tag{8}$$

where $Z(\epsilon)\exp\{g^2[\phi^{(+)}(x+\epsilon/2), \phi^{(-)}(x-\epsilon/2)]\} = \exp\{-ig^2D^{(+)}(\epsilon)\}$ and $D^{(+)}(x-y)$ is the corresponding two-point function. Applying the point-splitting regularization to the interacting current, we find

$$\begin{aligned} J^\mu(x) &= s\lim_{\epsilon\to 0}\frac{1}{2}\left\{Z(\epsilon)\overline{\psi}\left(x+\frac{\epsilon}{2}\right)e^{ig\phi^{(-)}(x+\frac{\epsilon}{2})}e^{ig\phi^{(+)}(x+\frac{\epsilon}{2})}\right. \\ &\quad \left.\times\gamma^\mu e^{-ig\phi^{(-)}(x-\frac{\epsilon}{2})}e^{-ig\phi^{(+)}(x-\frac{\epsilon}{2})}\psi\left(x-\frac{\epsilon}{2}\right) + \text{H.c.}\right\} \\ &= :\overline{\psi}(x)\gamma^\mu\psi(x): + \frac{g}{2\pi}\partial^\mu\phi(x). \end{aligned} \tag{9}$$

Here $s\lim$ designates the symmetric limit, H.c. means Hermite conjugate and we have used the free-field relation $\overline{\psi}(x+\epsilon/2)\gamma^\mu\psi(x-\epsilon/2) = :\overline{\psi}(x)\gamma^\mu\psi(x): - \frac{i}{\pi}\frac{\epsilon^\mu}{\epsilon^2}$. Note that all singular terms have been automatically canceled in (9) due to the manifestly hermitian definition of the current, so that no vacuum subtractions are needed. The constant $Z(\epsilon)$ got canceled by the factor $Z^{-1}(\epsilon)$ coming from normal ordering of the two exponentials sandwiching γ^μ in (9). The quantum current $J^\mu(x)$ is not conserved (it is "anomalous"), $\partial_\mu J^\mu(x) = \frac{g}{2\pi}\Box\phi(x)$. However, it is obvious that the only effect of the anomaly is to renormalize the scalar field mass,

$$\partial_\mu\partial^\mu\phi + \tilde{\mu}^2\phi = 0, \quad \tilde{\mu}^2 = \frac{\mu^2}{1-\frac{g^2}{2\pi}}. \tag{10}$$

An analogous calculation of the quantum axial vector current yields

$$J_5^\mu(x) = :\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x): - \frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x), \tag{11}$$

which is a conserved quantity (due to the conservation of its free part and the presence of $\epsilon^{\mu\nu}=-\epsilon^{\nu\mu}$).

The conjugate momenta $\Pi_\phi = \partial_0\phi(x) - gJ^0$, $\Pi_\Psi = \frac{i}{2}\Psi^\dagger$, $\Pi_{\Psi^\dagger} = -\frac{i}{2}\Psi$ lead from the Lagrangian (1) to the Hamiltonian $H = H_{0B} + H'$. H_{0B} corresponds to the free massive scalar field and