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Knot topology in QCD

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ABSTRACT

We consider topological structure of classical vacuum solutions in quantum chromodynamics. Topologically non-equivalent vacuum configurations are classified by non-trivial second and third homotopy groups for coset of the color group $SU(N)$ $(N = 2, 3)$ under the action of maximal Abelian stability group. Starting with explicit vacuum knot configurations we study possible exact classical solutions. Exact analytic non-static knot solution in a simple CP^1 model in Euclidean space–time has been obtained. We construct an ansatz based on knot and monopole topological vacuum structure for searching new solutions in $SU(2)$ and $SU(3)$ QCD. We show that singular knot-like solutions in QCD in Minkowski space–time can be naturally obtained from knot solitons in integrable CP^1 models. A family of Skyrme type low energy effective theories of QCD admitting exact analytic solutions with non-vanishing Hopf charge is proposed.

1. Introduction

Topological structure of classical solutions in $SU(N)$ Yang–Mills theory implies numerous physical manifestations in such important phenomena in quantum chromodynamics (QCD) as the chiral symmetry breaking and confinement [1]. The most attractive mechanism of the confinement is based on the Meissner effect in dual color superconductor [2–4] where monopole vacuum condensate is generated dynamically due to quantum corrections [5–7]. Dyons represent alternative topological defects which may play an important role as well as monopoles in description of the confinement at zero and finite temperature [8]. Whereas the instanton and monopole solutions correspond to non-trivial topological Chern–Simons and monopole charges, the topological knot configurations with a non-zero Hopf charge represent another topological objects which become essential in various applications in standard QCD and effective Skyrme type theories of QCD in low energy region [9,10]. It has been found that knot solitons could be good candidates for description of glueball states which can be treated as excitations over the condensed vacuum [11,12].

The rich topological structure of QCD as a gauge theory is conditioned by the presence of non-trivial homotopy groups $\pi_k(SU(N)/H)$, where the stability subgroup H determines possible coset spaces with different topological properties. The homotopy group $\pi_3(SU(N)) = Z$ describes topological classes of instanton field configurations corresponding to topological Pontryagin index [13–15]. It is well known that instantons realize tunneling between topologically non-equivalent vacuums and provide dominant contribution to chiral symmetry breaking. Another example of manifestation of non-trivial topology in QCD is provided by the second homotopy group $\pi_2(SU(3)/U(1) \times U(1)) = Z \times Z$ which implies Weyl symmetric structure of vacuum and singular monopole solutions in $SU(3)$ QCD [16–19].

A nice feature of quantum chromodynamics is that gauge connection (potential) allows natural implementation of the color vector $\hat{n}$ in adjoint representation of $SU(N)$ within the formalism of gauge invariant Abelian decomposition suggested first in [17, 20–22] and developed further in [23–25]. The color vector $\hat{n}$ corresponds to generators of the Cartan subalgebra of Lie algebra $su(N)$ and gives a suitable tool for description of whole topological structure of the gauge theory. A crucial observation has been made that the classical vacuum in QCD can be explicitly constructed in terms of the color vector $\hat{n}$ [26,27]. *This immediately implies that classical vacuum is strongly degenerated and all topologically non-equivalent vacuums are classified by non-trivial homotopy groups* $\pi_{2,3}(SU(N)/H)$. In particular, in the case of $SU(3)$ QCD it has been shown that classical vacuum possesses a non-trivial Weyl symmetric structure described by the second homotopy group $\pi_2(SU(3)/U(1) \times U(1))$ [19]. It should be stressed, that the color vector $\hat{n}$ represents pure

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topological degrees of freedom, so that we have still a standard QCD. In low energy region, in effective QCD theories like a generalized Faddeev–Skyrme model [11], the vector $\hat{n}$ becomes dynamical. The knowledge of the classical vacuum structure allows to study vacuum excitations in search of possible finite energy topological solutions and make further steps towards understanding fundamental properties of QCD at quantum level.

In the present Letter we consider first the topological structure of the classical vacuum in $SU(N)$ ($N=2,3$) QCD and study possible manifestations of topological properties related with the homotopy groups $\pi_{2,3}(SU(3)/U(1)\times U(1))$ and $\pi_{2,3}(SU(2)/U(1))$. Starting with known exact knot solutions from the integrable sector of CP^1 models [28,29] we will construct a vacuum with knot topology and obtain new analytic classical solutions in CP^1 model, standard QCD (in Euclidean and Minkowski space–time) and effective Skyrme type theory. The Letter is organized as follows. In Section 2 we describe the general topological structure of the classical vacuum in $SU(2)$ and $SU(3)$ QCD. In Section 3 we consider a simple CP^1 model which can be treated as a restricted QCD with one field variable $\hat{n}$. Exact non-static knot-like solution with a finite Euclidean action has been found. Section 4 deals with an ansatz for possible topological solutions based on classical vacuum made of $\hat{n}$ with general topology. In Section 5 we present analytic singular knot-like solutions in QCD in Minkowski space–time. A family of generalized Skyrme type effective theories admitting exact solutions with non-trivial Hopf numbers is proposed in Section 6.

2. Topological structure of classical vacuum in QCD

2.1. Topology of $SU(2)$ QCD vacuum

It has been shown that knot configurations providing minimums of the energy functional in Faddeev–Skyrme model may correspond to vacuums of QCD in maximal Abelian gauge [26]. Later it has been proved that topologically non-equivalent classical vacuums in pure QCD can be constructed explicitly in terms of a color vector $\hat{n}^a$ ($a=1,2,3$) [27]. By this, the vacuum pure gauge fields $\vec{A}_\mu$ with different Chern–Simons numbers are in one-to-one correspondence with color fields $\hat{n}$ of respective Hopf charges.

One should stress, that the color vector $\hat{n}$ in CP^1 models (as well as in Faddeev–Skyrme theory) represents dynamic field variable whereas in QCD the vector field $\hat{n}$ contains only pure topological degrees of freedom. The most appropriate way how to implement the topological degrees of freedom of $\hat{n}$ into the gauge potential while keeping a standard QCD theory is provided by Cho–Duan–Ge gauge invariant Abelian projection [17,20,21]

$$\vec{A}_\mu = A_\mu\hat{n} + \vec{C}_\mu + \vec{X}_\mu \equiv \hat{A}_\mu + \vec{X}_\mu, \tag{1}$$

where A_μ and $\vec{X}_\mu$ are the Abelian and off-diagonal gauge potentials, $\hat{A}_\mu$ is a restricted part of the gauge potential, and $\vec{C}_\mu \equiv -\frac{1}{g}\hat{n}\times\partial_\mu\hat{n}$ is a magnetic potential. For simplicity we put the coupling constant g equal to one. The vector $\hat{n}$ has a natural origin in the mathematical structure of the gauge theory, it is defined on the coset G/H where the stability group H is defined by Cartan subalgebra generators of the Lie algebra $\mathfrak{g}(G)$. Notice, that there is another type of Abelian decomposition proposed in [23–25] which treats the color vector $\hat{n}$ as a part of the whole gauge potential. So that, such a decomposition leads to a theory different from the standard QCD already at classical level [30].

The magnetic field strength $\vec{H}_{\mu\nu}$ constructed from the magnetic gauge potential $\vec{C}_\mu$ defines the scalar magnetic field $H_{\mu\nu}$

$$\vec{H}_{\mu\nu} = \partial_\mu\vec{C}_\nu - \partial_\nu\vec{C}_\mu + \vec{C}_\mu\times\vec{C}_\nu \equiv H_{\mu\nu}\hat{n}. \tag{2}$$

The magnetic field $H_{\mu\nu}$ defines a closed differential 2-form $H = dx^\mu\wedge dx^\nu H_{\mu\nu}$ which implies the existence of dual magnetic potential $\tilde{C}_\mu$

$$H_{\mu\nu} = \partial_\mu\tilde{C}_\nu - \partial_\nu\tilde{C}_\mu. \tag{3}$$

An explicit construction of the classical vacuum of QCD in terms of knot configurations of $\hat{n}$ has been found first in [27]

$$\vec{A}^{vac}_\mu = -\tilde{C}_\mu\hat{n} + \vec{C}_\mu. \tag{4}$$

This relation establishes connection between a pure gauge potential and color vector field $\hat{n}$ and implies that the classical vacuum configurations can be described by topologically non-equivalent classes of the color field $\hat{n}$. Namely, the topological classes of $\hat{n}$ are determined by two homotopy groups, $\pi_2(SU(2)/U(1))$ and $\pi_3(SU(2)/U(1)) = \pi_3(S^2)$. The first one describes monopole configurations, whereas the second homotopy describes Hopf mapping $\hat{n}: S^3\to S^2$ (we assume that the space R^3 is compactified to a three-dimensional sphere S^3). So that, all topological non-equivalent classical vacuums are classified by Hopf, Q_H, and monopole, g_m, charges

$$Q_H = \frac{1}{32\pi^2}\int d^3x\epsilon^{ijk}\tilde{C}_i H_{jk},$$
$$g_m = \int_{S^2}\vec{H}_{ij}\cdot\hat{n}\,d\sigma^{ij}. \tag{5}$$

One can show that Hopf number equals to the Chern–Simons number for vacuum gauge field configurations A^{vac}_μ constructed from $\hat{n}$.

To study possible exact solutions in QCD and QCD effective theories we will consider explicit expressions for the color vector $\hat{n}$ with a given knot topology. In particular, we will use known exact analytic knot solutions found in special integrable models. Let us recall first an explicit construction of a simple knot configuration of $\hat{n}$ as a mapping $S^3\to S^2$ with unit Hopf charge. Surprisingly, such a simple construction leads directly to exact knot solutions found in CP^1 integrable models. Using stereographic projection it is convenient to parameterize the target space S^2 by a complex field $u\in C^1$

$$\hat{n} = \frac{1}{1+uu^*}\begin{pmatrix} u+u^* \\ -i(u-u^*) \\ uu^*-1\end{pmatrix}. \tag{6}$$

A three-dimensional sphere S^3 is given by embedding into R^4 as follows

$$|z_1|^2+|z_2|^2 = 1 \tag{7}$$

where z_1, z_2 are complex coordinates on the complex plane C^2. The Hopf mapping with the Hopf charge $Q_H = 1$ is determined by the following equation

$$u = \frac{z_1}{z_2}. \tag{8}$$

Starting with a given color vector $\hat{n}$ one can define the magnetic field $H_{\mu\nu}$ explicitly in terms of the complex field u

$$H_{\mu\nu} = \epsilon^{abc}\hat{n}^a\partial_\mu\hat{n}^b\partial_\nu\hat{n}^c$$
$$= \frac{-2i}{(1+|u|^2)^2}\left(\partial_\mu u\partial_\nu u^* - \partial_\nu u\partial_\mu u^*\right). \tag{9}$$

The dual magnetic potential $\tilde{C}_\mu$, (3) is written through the complex $SU(2)$ doublet $\zeta = (z_1, z_2)$ as follows

$$\tilde{C}_\mu = -2i\zeta^\dagger\partial_\mu\zeta. \tag{10}$$