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A spin-foam vertex amplitude with the correct semiclassical limit

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ABSTRACT

Spin-foam models are hoped to provide a dynamics for loop quantum gravity. These start from the Plebanski formulation of gravity, in which gravity is obtained from a topological field theory, BF theory, through constraints, which, however, select more than one gravitational sector, as well as an unphysical degenerate sector. We show this is why terms beyond the needed Feynman-prescribed one appear in the semiclassical limit of the EPRL spin-foam amplitude. By quantum mechanically isolating a single gravitational sector, we modify this amplitude, yielding a spin-foam amplitude for loop quantum gravity with the correct semiclassical limit.

1. Introduction

Loop quantum gravity (LQG) [1–4] offers a compelling kinematical framework in which discreteness of geometry is *derived* from a quantization of general relativity (GR) rather than postulated. The discreteness has enabled well-defined proposals for the Hamiltonian constraint — defining the dynamics of the theory — in which one sees how diffeomorphism invariance eliminates normally problematic ultraviolet divergences. However, the lack of manifest space–time covariance, inherent in any canonical approach, is often suspected as a reason for the presence of ambiguities in the quantization of the Hamiltonian constraint. This has motivated the *spin-foam* program [1,5–7], which aims to provide a space–time covariant, path integral version of the dynamics of LQG a la Feynman. The histories summed over in the path integral arise from loop quantization methods, each representing a 'quantum space–time', and referred to as a *spin-foam*.

At the heart of the path integral approach is the prescription that the contribution to the transition amplitude by each history should be the exponential of i times the action. The use of such an expression has roots tracing back to Paul Dirac's *Principles of Quantum Mechanics* [8], and is central to the successful derivation of the classical limit of the path integral. In spin-foams, the 'quantum space–times' have a classical geometric interpretation only in the semiclassical limit $\hbar \to 0$. It is in this limit that one seeks a spin-foam amplitude equal to the exponential of i times the classical action. We call this the 'semiclassical limit' of a spin-foam amplitude, following [9]. As highlighted in these remarks, having such a correct semiclassical limit is key in recovering the correct *classical* limit of the theory in the standard way.

The *method* used for constructing the individual amplitudes in a spin-foam sum is to use the *Plebanski* formulation of gravity, or variations thereof. In this formulation of gravity, one takes advantage of the fact that GR can be formulated as a topological field theory whose spin-foam quantization is well-understood — BF theory [10] — supplemented by so-called *simplicity* constraints. Within the last several years, a spin-foam model of quantum gravity was, for the first time, introduced whose kinematics match those of LQG and therefore realize the original goal of the spin-foam program: to provide a path integral dynamics for LQG. This is known in the literature as EPRL [11–14]; when the Barbero–Immirzi parameter [15,16] γ, a certain quantization ambiguity, is less than 1, this model is identical to the Freidel–Krasnov model [17]. Despite its success, the EPRL amplitude still has difficulty in obtaining the correct semiclassical limit: (non-geometric) degenerate configurations are not suppressed, and even if one restricts to non-degenerate configurations, the semiclassical limit of the simplest component of the amplitude, the vertex amplitude, has four terms instead of the desired one term of the form exponential of i times the action [18]. Both of these problems cause unphysical configurations to dominate in the semiclassical limit, as we will show. (See also additional arguments [19–22] on the importance of having only the one exponential term, reviewed in the final discussion.) Furthermore, we will show that both of these problems are directly due to a deficiency in the way gravity is recovered from BF theory: When one imposes the simplicity constraints, one isolates not just a single gravitational sector, but multiple sectors, not all physical. The other 4-d spin-foam models of gravity have similar problems with a similar source [9,23,24].

In the present work, we show how, by formulating the restriction to what we call the Einstein–Hilbert sector classically first, quantizing it, and incorporating it into the EPRL vertex definition, one can define a modified vertex for which the extra terms in

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the semiclassical limit are eliminated, degenerate configurations are exponentially suppressed, and one achieves a vertex amplitude with *the correct semiclassical limit*. This new modified vertex, which we call the proper EPRL vertex, additionally continues to be compatible with loop quantum gravity, linear in the boundary state, and $SU(2)$ invariant. The key condition of linearity in the boundary state ensures that the final transition amplitude defined by the spin-foam model is linear in the initial state and anti-linear in the final state.

To begin, we review the classical discrete framework, review the EPRL vertex, point out its problems, and then derive the solution, leading to the definition of the proper EPRL vertex. In the final discussion, we note how the proper vertex may also solve other problems in the literature, in addition to the above one originally motivating this work. This Letter provides a summary of the work, with emphasis on motivation and broader consequences, detailed proofs being left to the two longer articles [25,26].

2. A review of EPRL from a new perspective

The quantum histories used in spin-foam sums are usually based on a triangulation of space–time into 4-simplices. The probability amplitude for a given spin-foam history breaks up into a product of amplitudes associated to each component of the triangulation [1,7]. The most important of these amplitudes is the *vertex amplitude*, which provides the probability amplitude for data associated to a single 4-simplex. In the following, as we are concerned specifically with the vertex amplitude, for conceptual clarity, we focus on a single 4-simplex σ. (Though the EPRL vertex has been generalized to arbitrary cells [27], we restrict ourselves to the simplicial case, as certain key elements will depend on the combinatorics of this case. See final discussion.) Let triangles and tetrahedra of σ be denoted respectively by f and t and decorations thereof. Fix a transverse orientation of each f within the boundary of σ. Furthermore, fix an affine structure, which is equivalent to fixing a flat connection ∂_a, on σ; this is a pure gauge choice [26]. The basic variables for the single 4-simplex σ consist in 5 group elements $(G_t \in Spin(4))_{t\in\sigma}$, and 20 algebra elements $(B^{IJ}_{tf} \in \mathfrak{so}(4))_{f\in t\in\sigma}$, $I, J = 0,1,2,3$. These are subject to constraints: (1.) 'orientation', $G_t \triangleright B_{ft} = -G_{t'} \triangleright B_{ft'}$, where $\triangleright$ denotes adjoint action, (2.) 'closure', $\sum_{f\in t} B_{ft} = 0$, and (3.) 'linear simplicity', $(B_{ft})^{ij} = 0$, $i, j = 1,2,3$. Each of these three constraints either restrict the allowed histories in the spin-foam sum or are imposed in the sense that violations are exponentially suppressed. Constraints (1.) and (2.) imply [25,28] that there exists a *unique two-form* $B^{IJ}_{\mu\nu}$, constant with respect to ∂_a, such that, for all t, f with $f \in t$,

$$G_t \triangleright B^{IJ}_{ft} = \int_f B^{IJ}. \tag{1}$$

In this Letter, $\mu, \nu, \ldots$ denote tensor indices over σ as a manifold. When the constraint (3.), linear simplicity, is additionally imposed, $B^{IJ}_{\mu\nu}$ takes one of the three forms [25]

$$\text{(II}\pm\text{)}\ B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L \quad \text{for some const. } e^I_\mu,$$

$$\text{(deg)}\ \epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = 0 \quad \text{(degenerate } B\text{)}, \tag{2}$$

where ϵ_{IJKL} is the Levi-Civita array, and the names for these sectors have been taken from [25,29]. In sectors (II+) and (II−), e^I_μ has the interpretation of a *co-tetrad*, determining the space–time metric via $g_{\mu\nu} := \eta_{IJ} e^I_\mu e^J_\nu$, where $\eta_{IJ} := \mathrm{diag}(-1,1,1,1)$. Note that, despite the spatial indices ij appearing in the constraint (3.), the e^I_μ arising in this way has full $SO(4)$ freedom intact: For all $H \in Spin(4)$, under $G_t \mapsto HG_t$, we have $e^I_\mu \mapsto H^I{}_J e^J_\mu$ where $H^I{}_J$ is the $SO(4)$ matrix canonically associated to H (see, e.g., [18,26]).

If $B^{IJ}_{\mu\nu}$ is non-degenerate, it additionally defines a dynamically determined orientation of σ, which we represent by its sign relative to the fixed orientation $\mathring{\epsilon}$ of σ:

$$\omega := \mathrm{sgn}\big(\mathring{\epsilon}^{\mu\nu\rho\sigma}\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma}\big).$$

For convenience, define $\omega = 0$ when $B^{IJ}_{\mu\nu}$ is degenerate. Additionally, let $\nu := \pm 1, 0$ according to whether $B^{IJ}_{\mu\nu}$ is in (II±) or (deg). If $\nu \neq 0$, the BF Lagrangian is related to the Einstein–Hilbert Lagrangian by

$$\mathcal{L}_{BF} = \omega\nu\mathcal{L}_{EH}.$$

When $\omega\nu = +1$, $\mathcal{L}_{BF} = \mathcal{L}_{EH}$ and we say that $B^{IJ}_{\mu\nu}$, and the data (B^{IJ}_{ft}, G_t) determining $B^{IJ}_{\mu\nu}$, are in the *Einstein–Hilbert sector*.

What we have described until now are the discrete *space–time* variables of the model. These determine the *phase space* variables (G_f, J^{IJ}_{ft}) on the boundary via

$$G_f := G_{t'_f t_f} := G^{-1}_{t'_f} G_{t_f} \in Spin(4),$$

$$J^{IJ}_{ft} := \frac{1}{8\pi G}\left(B^{IJ}_{ft} + \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL} B^{IJ}_{ft}\right).$$

Here t_f, t'_f are respectively the tetrahedron 'above' and 'below' f within the boundary $\partial\sigma$ of σ. The J^{IJ}_{ft} are conjugate to the G_f in the sense that they generate left or right translations on G_f depending on whether $t = t_f$ or $t = t'_f$. The generators of (internal) spatial rotations in terms of these are then $L^i_{ft} := \frac{1}{2}\epsilon^i{}_{jk} J^{jk}_{ft}$.

In quantum theory, the simplicity constraint reduces the boundary Hilbert space of the quantum BF theory to that of LQG, yielding an embedding of LQG boundary states into $Spin(4)$ BF theory boundary states [14]. Let us recall this embedding both because it is at the heart of the EPRL vertex amplitude, and because it will be key in the modification we propose.

The LQG Hilbert space associated to $\partial\sigma$ is $L^2(\times_f SU(2))$. A (generalized) *spin-network* $\Psi_{(k_f,\psi_{ft})}$ in this space is labeled by one spin k_f and two states $\psi_{ft'_f} \in V^*_{k_f}$, $\psi_{ft_f} \in V_{k_f}$ per triangle f, where V_k denotes the spin-k representation of $SU(2)$. $\Psi_{(k_f,\psi_{ft})} \in L^2(\times_f SU(2))$ is given explicitly by

$$\Psi_{(k_f,\psi_{ft})}((g_f)) := \prod_f \langle \psi_{ft'_f} | \rho(g_f) | \psi_{ft_f} \rangle, \tag{3}$$

where $\rho(g)$ denotes the action of $g \in SU(2)$. The embedding ι from LQG states to $Spin(4)$ BF theory boundary states is defined in terms of the basis (3) by

$$(\iota\Psi_{(k_f,\psi_{ft})})((G_f)) := \prod_f \langle \psi_{ft'_f} | \iota^{k_f} \rho(G_f) \iota_{k_f} | \psi_{ft_f} \rangle,$$

where here and throughout this Letter we set $s^\pm := \frac{1}{2}|1 \pm \gamma| k$, $\iota_k : V_k \to V_{s^-} \otimes V_{s^+}$ denotes the intertwiner among the indicated $SU(2)$ representations, scaled such that it is isometric in the Hilbert space inner products, $\iota^k : V_{s^-} \otimes V_{s^+} \to V_k$ denotes its Hermitian conjugate, and $\rho(G)$ denotes the action of $G \in Spin(4)$ in the appropriate representation. Note that in order to ensure that $s^\pm_f$ are half integers, the values of k_f must be restricted; the resulting spectra of geometric operators then become continuous in the semiclassical limit if and only if γ is rational, so that γ must be rational in order for the theory to be viable [14,30]. (This is an