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Exact quantum-statistical dynamics of time-dependent generalized oscillators

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ABSTRACT

Using a pair of linear invariant operators satisfying the Liouville–von Neumann equation, we find the most general thermal density operator and Wigner function for time-dependent generalized oscillators. The general Wigner function has five free parameters and describes the thermal Wigner function about a classical trajectory in phase space. The contour of the Wigner function depicts an elliptical orbit with a constant area moving about the classical trajectory, whose eccentricity determines the squeezing of the initial vacuum.

A quantum system of time-dependent oscillators has been a continuing issue of interest since the advent of quantum mechanics. Paul trap is one of such oscillators, which has a time-periodic frequency [1]. The time-dependent generalized oscillators have been used to study Berry's phase [2,3] and the nonadiabtic phase [4–6]. Various methods have been applied to time-dependent quantum oscillators in many areas [7,8]. Agarwal and Kumar, and Aliaga et al. studied statistical properties of time-dependent oscillators [9,10]. Also the density matrix and density operator for time-dependent oscillators were studied in Refs. [11,12].

On the other hand, Lewis and Riesenfeld [13] introduced a method to find the exact quantum states for the time-dependent Schrödinger equation. In particular, for time-dependent oscillators they found a quadratic invariant operator, satisfying the quantum Liouville–von Neumann equation, whose eigenstates provide the exact quantum states up to time-dependent phase factors. Each complex solution to the classical equation of motion leads to a pair of invariant operators, linear in position and momentum operators, for time-dependent oscillators [14] and even for time-dependent generalized oscillators [15].

In this Letter, using the linear invariant operators which satisfy the Liouville–von Neumann equation, we find in a constructive way the most general thermal density operator and Wigner function up to the quadratic order in position and momentum operators. This density operator is a squeezed and displaced state of a thermal one. Further, the density matrix is the thermal one shifted by a real classical solution, which has five free parameters. The contour of the Wigner function follows an elliptical orbit with a constant area whose center moves and principal axes rotate along a classical trajectory. The shape of the ellipse measured by eccentricity determines the squeezing of the initial vacuum.

The time-dependent generalized quantum oscillator is described by the Hamiltonian [2,3,15–17]

$$\hat{H}(t) = \frac{X(t)}{2}\hat{p}^2 + \frac{Y(t)}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) + \frac{Z(t)}{2}\hat{q}^2, \tag{1}$$

where X, Y and Z explicitly depend on time. Lewis and Riesenfeld have shown that the invariant operator satisfying the quantum Liouville–von Neumann equation

$$i\hbar\frac{\partial}{\partial t}\hat{I}(t) + \left[\hat{I}(t), \hat{H}(t)\right] = 0, \tag{2}$$

provides the exact quantum states of the time-dependent Schrödinger equation as its eigenstates up to time-dependent phase factors. Following Ref. [15], we introduce a pair of linear invariant operators for Eq. (2)

$$\hat{a}_u(t) = \frac{i}{\sqrt{\hbar}}\left[u^*(t)\hat{p} - \frac{1}{X(t)}\left[\dot{u}^*(t) - Y(t)u^*(t)\right]\hat{q}\right],$$
$$\hat{a}_u^\dagger(t) = -\frac{i}{\sqrt{\hbar}}\left[u(t)\hat{p} - \frac{1}{X(t)}\left[\dot{u}(t) - Y(t)u(t)\right]\hat{q}\right], \tag{3}$$

where u is a complex solution to the classical equation of motion

$$\frac{d}{dt}\left(\frac{\dot{u}}{X}\right) + \left[XZ - Y^2 + \frac{\dot{X}Y - X\dot{Y}}{X}\right]\left(\frac{u}{X}\right) = 0, \tag{4}$$

with overdots denoting the derivative with respect to t. Normalizing the complex solution to satisfy the Wronskian condition

$$\mathrm{Wr}\{u, u^*\} = \frac{1}{X}(u\dot{u}^* - u^*\dot{u}) = i, \tag{5}$$

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one can make the invariant operators satisfy the standard commutation relation at equal time

$$\big[\hat{a}_u(t), \hat{a}_u^\dagger(t)\big] = 1. \tag{6}$$

Another complex solution v to Eq. (4), which can be expressed as

$$v(t) = \mu^* u(t) - \nu^* u^*(t) \tag{7}$$

for complex constants μ and ν given by

$$\mu = i\,\mathrm{Wr}\big\{v^*, u\big\}, \qquad \nu = i\,\mathrm{Wr}\big\{v^*, u^*\big\}, \tag{8}$$

leads to another set of the invariant operators $\hat{a}_v(t)$ and $\hat{a}_v^\dagger(t)$. Now the Wronskian condition on v

$$\mathrm{Wr}\big\{v, v^*\big\} = i \quad \Leftrightarrow \quad |\mu|^2 - |\nu|^2 = 1, \tag{9}$$

also guarantees the standard commutation relation at equal time

$$\big[\hat{a}_v(t), \hat{a}_v^\dagger(t)\big] = 1. \tag{10}$$

In fact, these two sets of invariant operators are related through the Bogoliubov transformation

$$\begin{aligned} \hat{a}_v(t) &= \mu \hat{a}_u(t) + \nu \hat{a}_u^\dagger(t), \\ \hat{a}_v^\dagger(t) &= \mu^* \hat{a}_u^\dagger(t) + \nu^* \hat{a}_u(t). \end{aligned} \tag{11}$$

The Bogoliubov transformation is written as the similarity transform [9,18]

$$\hat{a}_v(t) = \hat{S}^{-1}(t)\hat{a}_u(t)\hat{S}(t), \qquad \hat{a}_v^\dagger(t) = \hat{S}^{-1}(t)\hat{a}_u^\dagger(t)\hat{S}(t), \tag{12}$$

by the squeezing operator

$$\begin{aligned} \hat{S}(t) = e^{i\theta_\mu \hat{a}_u^\dagger(t)\hat{a}_u(t)} \exp\bigg[&\frac{1}{2}e^{i(\theta_\nu - \theta_\mu)} \cosh^{-1}|\mu| \hat{a}_u^{\dagger 2}(t) \\ &- \frac{1}{2}e^{-i(\theta_\nu - \theta_\mu)} \cosh^{-1}|\mu| \hat{a}_u^2(t)\bigg], \end{aligned} \tag{13}$$

where $\mu = |\mu|e^{i\theta_\mu}$ and $\nu = |\nu|e^{i\theta_\nu}$. We may use the freedom in choosing the overall constant phase of $\hat{a}_v(t)$, which is not physically important, to fix the phase $\theta_\mu = 0$ [18]. Thus there are only two parameters $|\mu|$ and θ_ν or a complex constant ν, i.e. $|\nu|$ and θ_ν.

The most general, quadratic, Hermitian invariant operator constructed from the pair $\hat{a}_v(t)$ and $\hat{a}_v^\dagger(t)$ takes the form

$$\begin{aligned} \hat{\mathcal{I}}_v(t) = {}& \frac{A}{2}\hat{a}_v^{\dagger 2}(t) + \frac{B}{2}\big[\hat{a}_v^\dagger(t)\hat{a}_v(t) + \hat{a}_v(t)\hat{a}_v^\dagger(t)\big] \\ &+ \frac{A^*}{2}\hat{a}_v^2(t) + D\hat{a}_v^\dagger(t) + D^*\hat{a}_v(t) + E, \end{aligned} \tag{14}$$

where A and D are complex constants, and B and E are real constants. By choosing μ and ν, i.e. u such that

$$A\mu^{*2} + 2B\mu^*\nu + A^*\nu^2 = 0, \tag{15}$$

the invariant operator (14) can be written in the canonical form

$$\hat{\mathcal{I}}_u(t) = \hbar\omega_0 \hat{a}_u^\dagger(t)\hat{a}_u(t) + \delta\hat{a}_u^\dagger(t) + \delta^*\hat{a}_u(t) + \epsilon, \tag{16}$$

where

$$\begin{aligned} \hbar\omega_0 &= A\mu^*\nu^* + B\big(|\mu|^2 + |\nu|^2\big) + A^*\mu\nu, \\ \delta &= D\mu^* + D^*\nu, \\ \epsilon &= E + \frac{1}{2}\hbar\omega_0. \end{aligned} \tag{17}$$

Hence this implies that by allowing all the complex u's satisfying both Eqs. (4) and (5) the invariant operator (16) is general enough for our purpose. From now on we shall work on the Fock bases $\hat{a}_u(t)$ and $\hat{a}_u^\dagger(t)$ for all the complex u's and drop the subscript u.

Since the invariant operator (16) satisfies Eq. (2), we may use it to define the density operator [12]

$$\hat{\rho}(t) = \frac{1}{Z}e^{-\beta\hat{\mathcal{I}}_u(t)}. \tag{18}$$

Here β is a free parameter that may be identified with the inverse temperature of the system in equilibrium, and $Z = \mathrm{Tr}(e^{-\beta\hat{\mathcal{I}}_u}(t))$. The density operator has five free parameters, i.e. $\beta\omega_0$, a complex constant δ, which is related with the classical position q_c and momentum p_c as will be shown below, and $|\mu|$ and θ_ν in choosing u. By introducing the displacement operator

$$\hat{D}(z) = e^{-z\hat{a}^\dagger(t) + z^*\hat{a}(t)}, \tag{19}$$

with $z = -\delta/(\hbar\omega_0)$, $\epsilon = |\delta|^2/(\hbar\omega_0)$, we write the density operator as

$$\hat{\rho}(t) = \hat{D}^\dagger(z)\hat{\rho}_{\mathrm{T}}(t)\hat{D}(z), \tag{20}$$

where

$$\hat{\rho}_{\mathrm{T}}(t) = \frac{1}{Z_{\mathrm{T}}}e^{-\beta\hbar\omega_0\hat{a}^\dagger(t)\hat{a}(t)}, \tag{21}$$

is a thermal density operator. It follows that $Z = Z_{\mathrm{T}}$ due to the unitary transformation (20). The coherent state, defined as $\hat{a}(t)|z,t\rangle = z|z,t\rangle$, is also given by

$$|z,t\rangle = \hat{D}^\dagger(z)|0,t\rangle, \tag{22}$$

where $|0,t\rangle$ is the vacuum state that is annihilated by $\hat{a}(t)$. The position and momentum expectation values with respect to the coherent state are

$$\begin{aligned} \langle z,t|\hat{q}|z,t\rangle &= \sqrt{\hbar}\big(uz + u^*z^*\big) \equiv q_c, \\ \langle z,t|\hat{p}|z,t\rangle &= -\frac{Y}{X}q_c + \frac{\sqrt{\hbar}}{X}\big(\dot{u}z + \dot{u}^*z^*\big) \equiv p_c. \end{aligned} \tag{23}$$

These q_c and p_c satisfy the classical Hamilton equations

$$\begin{aligned} \dot{q}_c &= Xp_c + Yq_c, \\ \dot{p}_c &= -Yp_c - Zq_c. \end{aligned} \tag{24}$$

Now, from the definition of the thermal expectation value

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}\big[\hat{\mathcal{O}}\hat{\rho}(t)\big] = \mathrm{Tr}\big[\hat{D}(z)\hat{\mathcal{O}}\hat{D}^\dagger(z)\hat{\rho}_{\mathrm{T}}\big], \tag{25}$$

we find the expectation values of position and momentum operators

$$\langle\hat{q}\rangle = q_c, \qquad \langle\hat{p}\rangle = p_c, \tag{26}$$

and those of quadratic operators

$$\begin{aligned} \langle\hat{q}^2\rangle &= \hbar u^*u(1+2\bar{n}) + q_c^2, \\ \langle\hat{p}^2\rangle &= \frac{\hbar}{X^2}\big(\dot{u}^* - Yu^*\big)(\dot{u} - Yu)(1+2\bar{n}) + p_c^2, \\ \bigg\langle\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\bigg\rangle &= \frac{\hbar}{2X}\big[\big(\dot{u}^* - Yu^*\big)u + (\dot{u} - Yu)u^*\big](1+2\bar{n}) \\ &\quad + q_cp_c, \end{aligned} \tag{27}$$

where