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Singularities of the Casimir energy for quantum field theories with Lifshitz dimensions

V.K. Oikonomou

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

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ABSTRACT

We study the singularities that the Casimir energy of a scalar field in spacetimes with Lifshitz dimensions exhibits, and provide expressions of the energy in terms of multidimensional zeta functions for the massless case. Using the zeta-regularization method, we found that when the four dimensional spacetime has Lifshitz dimensions, then for specific values of the critical exponents, the Casimir energy is singular, in contrast to the non-Lifshitz case. Particularly we found that when the value of the critical exponent is $z=2$, the Casimir energy is singular, while for $z \geqslant 3$ the Casimir energy is regular. In addition, when flat extra dimensions are considered, the critical exponents of the Lifshitz dimensions affect drastically the Casimir energy, introducing singularities that are absent in the non-Lifshitz case. We also discuss the Casimir energy in the context of braneworld models and the perspective of Lifshitz dimensions in such framework.

1. Introduction

During the last decades, the Casimir effect has played a prominent role in various research areas of theoretical physics (see for example [1–8] and references therein). The first theoretical study was done by Casimir [1], in 1948, who predicted an attractive force between two neutral perfectly conducting parallel plates. The Casimir effect is a manifestation of the vacuum corresponding to a quantum field theory. Owing to the vacuum fluctuations of the electromagnetic quantum field, the parallel plates attract each other. Moreover, the boundaries alter the quantum field boundary conditions and as a result the plates interact. The geometry of the boundaries have a strong effect on the Casimir energy and Casimir force. The calculations of the Casimir energy and the corresponding force were generalized to include other quantum fields such as fermions, bosons and scalar fields making the Casimir effect study an important ingredient of many theoretical physics subjects such as string theory, cosmology etc. The technological applications of the Casimir force are of invaluable importance, for instance in nanotubes, nano-devices and generally in microelectronic engineering [9]. Indeed an attractive or repulsive Casimir force can lead to the instability or even destruction of such a micro-device. Hence, studying various geometrical and material configurations will enable us to have control over the Casimir force.

The Casimir effect was verified experimentally [10,11] rendering the physics of such studies very valuable due to the theoretical outcomes of these studies. The Casimir energy studies have been done for curved spacetimes, for topologically non-trivial backgrounds and for various geometrical configurations and physical setups [1–8]. The applications of such calculations are numerous, constraining even cosmological models. Moreover, various models are put in test, for example the size and the shape of extra dimensions are constrained from Casimir energy calculations [12,13]. In view of these applications, every consistent quantum field theory is severely constrained by Casimir energy and Casimir force measurements.

Lifshitz type quantum field theories [15–37] serve as Lorentz violating field theories with remarkable properties. These theories have their origin in condensed matter physics [38] where a Lifshitz critical point is defined as a point in a phase diagram where three phases of a condensed matter system meet. Condensed matter systems exhibiting a Lifshitz point have an intrinsic space anisotropy, which is quantified in terms of the existence of two different correlation lengths in the anisotropic space dimensions. Lifshitz quantum field theories have been studied in flat and curved background and, although being Lorentz violating theories, the renormalization of the various loop integrals is improved, with the last being the most appealing attribute of these gauge theories. In addition, the class of the renormalizable interactions is sufficiently enriched, since the Lifshitz type operators that appear in a Lagrangian are higher derivatives of the quantum fields with mass dimensions. Another appealing feature is that, within

E-mail addresses: voiko@physics.auth.gr, Vasilis.Oikonomou@mis.mpg.de.

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the Lifshitz theoretical framework, dynamical mass generation naturally occurs, as a consequence of the dimensionfull couplings of Lifshitz operators. We shall study a massless scalar field, in the context of Lifshitz type dimensions, but in two different situations.

Since the Casimir energy has proved to be a very important property of any quantum field theory, we shall compute the Casimir energy of a scalar field, in four dimensional spacetime with Lifshitz anisotropic extra dimensions and also the Casimir energy in four dimensions with anisotropic scaling between space and time. We extend the last framework in the case which each space dimension is anisotropic. What we are mainly interested in, is the singularity structure of the Casimir energy in all the aforementioned cases. We want to explore if the introduction of a Lifshitz anisotropy in space introduces any new singularities. Since the Casimir energy is a benchmark for any viable quantum field theory, such a study is very important for the intrinsic validity of any quantum field theory.

In this Letter we shall calculate the Casimir energy for a massless scalar field confined in a three dimensional box with Dirichlet boundaries. It will be assumed that some of the dimensions are Lifshitz in various setups. As we shall demonstrate, the quantum theory with critical exponent $z=2$ yields a singular Casimir energy even though we use the zeta-regularization method. This is in contrast to the non-Lifshitz case and also for Lifshitz quantum theories with $z\geqslant 3$, where the Casimir energy is finite. Hence, theories with $z=2$ are put into question. In addition we shall assume that the four dimensional spacetime has Lifshitz extra dimensions. In this framework the four dimensional Minkowski spacetime Lorentz invariance is protected, while each extra dimension is anisotropic. However this anisotropy has a radical impact on the field theory Lagrangian, since it allows a quite large number of higher order derivatives of the fields. Nevertheless, when the theory is assumed to "sit" on a Lifshitz point, the only derivative terms that survive are the higher order ones, since the lower order derivatives at the Lifshitz point become irrelevant. This construction is very similar to the original condensed matter construction and additionally it doesn't affect the four dimensional Lorentz invariance. As we shall see, extra dimensions bring about singular terms that are absent in the non-Lifshitz case. Finally, since the Kaluza–Klein theories with flat extra dimensions are not considered to be realistic, we shall comment on the calculation of the Casimir energy in Randall–Sundrum like spacetimes and more generally, in the context of braneworld scenario. We shall assume the spacetime is curved and that gravity is coupled conformally to the scalar field. Particularly, the spacetime will shall use is a generalization of AdS spacetime with Lifshitz scaling invariance. As we shall evince, the calculation of the scalar Casimir energy in such backgrounds is not an easy task, since the Einstein equations have no solutions of Lifshitz type.

This Letter is organized as follows: In Section 1 we compute the Casimir energy for four dimensional Minkowski spacetime with anisotropic scaling between space and time. The massless scalar field is considered to be confined in a box, and also satisfies Dirichlet boundary conditions on the boundaries. We also consider the case where the three space dimensions are anisotropic. In Section 2 we calculate the Casimir energy of a massless scalar field in a Minkowski spacetime background with a Lifshitz extra dimension and investigate how the singularities of the Casimir energy are affected by the anisotropic scaling of the extra dimension. Moreover we present the singularities of the massless scalar field Casimir energy, in Minkowski spacetime with two anisotropic extra dimensions. We discuss on the calculation of the Casimir energy in the context of braneworld scenario in Section 3. The conclusions with a discussion on the results follow in the end of this Letter.

2. Casimir energy for scalar field with three spatial Lifshitz dimensions

2.1. *Massless scalar case – identical scaling of the space dimensions*

We shall start with the study of the Casimir energy for a scalar field in three dimensional spacetime but with the space and time scaled differently. Particularly, we assume that the mass dimension of the spacetime coordinates is:

$$[t]=-z,\qquad [x_i]=-1 \tag{1}$$

with $i=1,2,3$. This case has been studied thoroughly and appears quite frequently in the literature, see for example Ref. [23]. Relation (1) stems from the Lifshitz scaling of the time coordinate in reference to that of the three spatial coordinates, namely:

$$x_i\to bx_i,\qquad t\to b^z t \tag{2}$$

As we mentioned in the introduction, although such a scaling is different in spirit regarding the original idea of Lifshitz [38], the renormalization properties of such scalar theories are remarkable rendering such theories really valuable, although Lorentz violating. We want to compute the Casimir energy in the case the system is confined in a box, with lengths, $0\leqslant x_i\leqslant L_i$ and $i=1,2,3$. Assuming relation (1), the action of the massless scalar field reads,

$$\mathcal{S}=\int dt\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} dx_1\,dx_2\,dx_3\,\Phi^*(t,x_i)$$
$$\times\Big(\big(\partial^t\big)^2-\big(-\partial_{x_1}^2\big)^z-\big(-\partial_{x_2}^2\big)^z-\big(-\partial_{x_3}^2\big)^z\Big)\Phi(t,x_i) \tag{3}$$

The scalar field is considered to obey Dirichlet boundary conditions at the boundary of the box,

$$\Phi(x_1,x_2,0)=\Phi(x_1,x_2,L_3)=0$$
$$\Phi(x_1,0,x_3)=\Phi(x_1,L_2,x_3)=0$$
$$\Phi(0,x_2,x_3)=\Phi(L_1,x_2,x_3)=0 \tag{4}$$

and as a consequence the Casimir energy is equal to:

$$\mathcal{E}_{z_1,z_2,z_3}=\sum_{n_1,n_2,n_3=1}^{\infty}\left[\left(\frac{2n_1\pi}{L_1}\right)^{2z}+\left(\frac{2n_2\pi}{L_2}\right)^{2z}+\left(\frac{2n_3\pi}{L_3}\right)^{2z}\right]^{-s} \tag{5}$$

The expression for the Casimir energy (5), can be written in terms of a multidimensional zeta function [14], namely in terms of the function,

$$\mathcal{M}_3(s_1;a_1,a_2,a_3;m_1,m_2,m_3)$$
$$=\sum_{n_1,n_2,n_3=1}^{\infty}\big(a_1n_1^{m_1}+a_2n_2^{m_2}+a_3n_3^{m_3}\big)^{-s_1}$$
$$\simeq\frac{1}{a_3^{s_1}\Gamma(s_1)}\left(\sum_{k_1,k_2=0}^{\infty}\frac{(-b_1)^{k_1}}{k_1!}\frac{(-b_2)^{k_2}}{k_2!}\Gamma(s_1+k_1+k_2)\right.$$
$$\left.\times\,\zeta(-m_1k_1)\zeta(-m_2k_2)\zeta\big(m_3(s_1+k_1+k_2)\big)\right)$$
$$+\frac{\Gamma(1/m_2)}{m_2b_2^{1/m_2}}\sum_{k_1=0}^{\infty}\frac{(-b_1)^{k_1}}{k_1!}$$
$$\times\,\Gamma(s_1+k_1-1/m_2)\zeta(-m_1k_1)\zeta\big(m_3(s_1+k_1-1/m_2)\big)$$