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Spectroscopy in finite density lattice field theory: An exploratory study in the relativistic Bose gas

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ABSTRACT

We analyze 2-point functions in the relativistic Bose gas on the lattice, i.e., a charged scalar ϕ^4 field with chemical potential μ. Using a generalized worm algorithm we perform a Monte Carlo simulation in a dual representation in terms of fluxes where the complex action problem is overcome. We explore various aspects of lattice spectroscopy at finite density and zero temperature, such as the asymmetry of forward and backward propagation in time and the transition into the condensed phase. It is shown that after a suitable subtraction the exponents for forward and backward propagation are independent of μ and agree with the mass obtained from the propagator at $\mu = 0$. This holds for $\mu < \mu_c$ and shows that below the condensation transition the mass is independent of μ as expected from the Silver Blaze scenario.

1. Introduction

In recent years several interesting developments for the simulation of lattice field theories at finite density were presented (for examples related to this work see, e.g., [1–8]). A key ingredient in these developments is a reformulation of the original lattice field theory in terms of new variables (dual variables) which usually are loops or surfaces. The dual variables are subject to constraints, but can be updated very efficiently with generalizations of the Prokof'ev–Svistunov worm algorithm [9]. Although some of the most interesting problems, such as finite density lattice QCD, are not yet under control, many interesting new concepts and techniques now enlarge our toolbox for such future challenges.

In this Letter we address the problem of spectroscopy in finite density lattice field theory. We study 2-point functions for a charged scalar ϕ^4 field with a chemical potential (relativistic Bose gas). We use a dual representation [8] in terms of fluxes where the partition function is a sum over closed loops of fluxes with real and positive weights, i.e., the complex action problem of the conventional representation is solved in the dual form. Our Monte Carlo update for the dual degrees of freedom is a generalization of the worm algorithm [9] and was presented in [8].

Lattice spectroscopy is based on 2-point functions where suitable monomials of the lattice fields are separated in Euclidean time. At finite density these two-point functions are modified since the chemical potential introduces an asymmetry in time: Matter which is propagating forward in time is enhanced by the chemical potential μ over anti-matter propagating backward in time. Thus one expects that finite μ introduces an asymmetry in 2-point functions. Furthermore, in finite density field theory condensation transitions can appear when the chemical potential exceeds a critical value μ_c (see [6,8] for an analysis of condensation in the relativistic Bose gas on the lattice). In this exploratory study we analyze the propagators in both phases, at $\mu < \mu_c$ where we study the asymmetry of the 2-point functions, and at $\mu > \mu_c$ where we observe condensation.

2. Conventional and dual formulation

Since later we will discuss also a continuum analysis of the charged scalar field, we start with presenting the action in the continuum form,

$$S = \int d^4x \big[|\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \lambda |\phi|^4 + \mu(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) \big]. \quad (1)$$

The chemical potential μ couples to the conserved charge of the scalar field, but also shifts the mass parameter. This has an important implication for the free theory ($\lambda = 0$): for $\mu > m$ the action is unbounded from below and the theory becomes unstable. For finite λ, sufficiently large chemical potential flips the sign of the quadratic term, giving rise to the Mexican hat potential and the transition into the condensed phase.

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In a lattice discretization the continuum action (1) assumes the form

$$S = \sum_x \Bigg(\eta |\phi_x|^2 + \lambda |\phi_x|^4 - \sum_{\nu=1}^{4} \big[e^{-\mu \delta_{\nu,4}} \phi_x^\star \phi_{x+\hat{\nu}} + e^{\mu \delta_{\nu,4}} \phi_x^\star \phi_{x-\hat{\nu}} \big] \Bigg). \tag{2}$$

The first sum is over the sites x of a $N_s^3 \times N_t$ lattice with periodic boundary conditions, the second sum is over the four Euclidean directions $\nu = 1, 2, 3, 4$, and $\hat{\nu}$ denotes the unit vector in ν-direction. The degrees of freedom are the complex valued field variables ϕ_x attached to the sites x of the lattice. η denotes the combination $8 + m^2$, where m is the bare mass parameter and the ϕ^4 coupling is denoted by λ. All parameters are in units of the lattice spacing a, in other words the lattice spacing is set to $a = 1$ throughout this Letter. The partition sum Z is obtained by integrating the Boltzmann factor e^{-S} over all field configurations, $Z = \int D[\phi] e^{-S}$. The measure is a product over all lattice sites x, with ϕ_x being integrated over the complex plane, i.e., $D[\phi] = \prod_x \int_{\mathbb{C}} d\phi_x / 2\pi$.

The lattice partition sum in the conventional form can be mapped exactly to a dual representation [8], where the new degrees of freedom are integer valued fluxes attached to the links. The dual partition sum is given by

$$Z = \sum_{\{k,l\}} \Bigg(\prod_x \delta \Bigg(\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}] \Bigg) \Bigg) \Bigg(\prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + l_{x,\nu})! \, l_{x,\nu}!} \Bigg) \times \Bigg(\prod_x e^{-\mu k_{x,4}} W \Bigg(\sum_\nu \big[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu}) \big] \Bigg) \Bigg). \tag{3}$$

The sum is over all configurations of two sets of flux variables: $k_{x,\nu} \in \mathbb{Z}$ and $l_{x,\nu} \in \mathbb{N}_0$. The k-fluxes are subject to a constraint enforced by the product over Kronecker deltas (for notational convenience here denoted by $\delta_{n,0} = \delta(n)$). The constraints enforce the conservation of k-flux at each site x, i.e., $\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}] = 0$ for all x. The l-fluxes are unconstrained. The configurations of k- and l-fluxes are weighted with various weights, one of them including the chemical potential, and the factors W are given by the integrals $W(n) = \int_0^\infty dr \, r^{n+1} e^{-\eta r^2 - \lambda r^4}$, which can be evaluated numerically to arbitrary precision. It is obvious that also at finite μ the weight factors are real and positive, and the complex action problem is solved in the dual representation.

For the Monte Carlo update we use a generalized worm algorithm [9] that was presented in [8]. In that Letter thermodynamical observables such as the particle number n, the field expectation value $\langle |\phi|^2 \rangle$, and the corresponding susceptibilities were studied. All these observables can be obtained as derivatives of $\ln Z$ with respect to the parameters μ and η and in the dual representation take the form of expectation values of moments of the flux variables. Based on these observables the phase diagram of the relativistic Bose gas and condensation phenomena were studied.

3. 2-point functions in the dual representation

Using continuum notation the 2-point function needed for spectroscopy is given by

$$C(t) = \int d^3 y \, \langle \phi(t, \vec{y}) \phi(0, \vec{0})^* \rangle \propto e^{-Et}, \tag{4}$$

where we project to zero spatial momentum, such that the energy E of the particle at rest, i.e., its mass, determines the exponential decay of $C(t)$. Thus, on the lattice we need to evaluate 2-point functions which in the conventional lattice representation are given by

$$\langle \phi_y \phi_z^* \rangle = \frac{1}{Z} \int D[\phi] \, e^{-S} \phi_y \phi_z^* \equiv \frac{1}{Z} Z_{y,z}. \tag{5}$$

Here $Z_{y,z}$ denotes the lattice partition sum with two field insertions $\phi_y \phi_z^*$. It is straightforward to generalize the derivation in [8] to find the dual representation of $Z_{y,z}$,

$$Z_{y,z} = \sum_{\{k,l\}} \Bigg(\prod_x \delta \Bigg(\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}] - \delta_{x,y} + \delta_{x,z} \Bigg) \Bigg) \times \Bigg(\prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + l_{x,\nu})! \, l_{x,\nu}!} \Bigg) \times \Bigg(\prod_x e^{-\mu k_{x,4}} W \Bigg(\sum_\nu \big[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu}) \big] + \delta_{x,y} + \delta_{x,z} \Bigg) \Bigg). \tag{6}$$

At the positions y and z where the source fields $\phi_y \phi_z^*$ are located the index of the weight factors $W(n)$ is increased (see the Kronecker deltas in the argument of the W). Also the constraints for the k-fluxes change to

$$\prod_x \delta \Bigg(\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}] - \delta_{x,y} + \delta_{x,z} \Bigg). \tag{7}$$

In the presence of the source fields $\phi_y \phi_z^*$ the usual local flux conservation $\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}] = 0$ at all sites x is modified such that the site z constitutes a source of flux, the site y a sink. Thus in the dual representation for $Z_{y,z}$ the set of configurations of allowed k-flux consists of closed loops and a single open line of flux connecting the sites y and z.

To evaluate $Z_{y,z}$ efficiently we follow the strategy of [5] and define a generalized partition function $\mathcal{Z}$ where we sum over all possible positions of the field insertions,

$$\mathcal{Z} = \sum_{u,v} Z_{u,v}. \tag{8}$$

The configurations that constitute $\mathcal{Z}$ consist of closed loops of flux plus a single flux line with open ends at arbitrary sites u, v of the lattice. For the Monte Carlo simulation of $\mathcal{Z}$ we can thus re-use the worm algorithm of [8], where now every step of the worm constitutes an admissible configuration of $\mathcal{Z}$ (not only the configurations where the worm has closed that contribute to Z). Thetwo-point functions are then obtained as

$$\langle \phi_y \phi_z^* \rangle = \frac{Z_{y,z}}{Z} = \frac{\langle \delta_{u,y} \delta_{v,z} \rangle_{\mathcal{Z}}}{\langle \delta_{u,v} W(f_u)/W(f_u + 2) \rangle_{\mathcal{Z}}}, \tag{9}$$

where $\langle \cdots \rangle_{\mathcal{Z}}$ denotes the expectation value with respect to the enlarged ensemble $\mathcal{Z}$. By $f_u = \sum_\nu [|k_{u,\nu}| + |k_{u-\hat{\nu},\nu}| + 2(l_{u,\nu} + l_{u-\hat{\nu},\nu})]$ we denote the combined k- and l-flux at site u that enters the weights W, and the reweighting with $W(f_u)/W(f_u + 2)$ in (9) is necessary to correctly obtain the original partition sum Z in the denominator of (9). When taking into account the projection to zero momentum we end up with the following dual representation of the correlator $C(t)$:

$$C(t) = \frac{\langle \delta_{t,u_4 - v_4} \rangle_{\mathcal{Z}}}{\langle \delta_{u,v} W(f_u)/W(f_u + 2) \rangle_{\mathcal{Z}}}. \tag{10}$$