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Does anomalous violation of null energy condition invalidate holographic c-theorem?

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ABSTRACT

Null energy condition plays a crucial role in holographic renormalization group flow, leading to the holographic c-theorem. Unfortunately, the null energy condition is quantum mechanically violated. Even the averaged version can be violated. We discuss how the anomalous violation of the null energy condition affects the holographic renormalization group flow in $(1+3)$-dimensional bulk gravity. We show that despite the violation of the null energy condition, a suitably modified holographic c-function with a peculiar log correction is still monotonically decreasing in so far as we add the counterterm that removes a ghost mode of gravity.

1. Introduction

The holographic renormalization is a beautiful scheme to geometrize the renormalization group flow. In the classical Einstein gravity approximation, the Weyl consistency condition of the local renormalization group is automatically realized as gravitational equations of motion. In particular, we can derive the holographic c-theorem [1–4] that dictates there exists a c-function that monotonically decreases along the renormalization group flow in any space–time dimensions.

In order to derive the holographic c-theorem, one crucial assumption is that the matter satisfies the null energy condition. This is reasonable at least classically because the field theoretic c-theorem, whose complete proof is available in $(1+1)$ dimensions [5] with recent significant progress in $(1+3)$ dimensions [6], requires the unitarity as a part of the assumption, and the null energy condition is naturally regarded as its gravitational counterpart.

In general relativity, any pathological space–time could be a solution of the equation of motion without assuming the energy condition because we can simply declare that the corresponding Einstein tensor is sourced by the same energy–momentum tensor. It is known that the null energy condition, reasonably satisfied in realistic classical matter systems, is sufficiently strong to avoid many "pathological" space–times such as wormhole, superluminal propagation, time-machine, shrinking black holes and so on.

Unfortunately (or fortunately), the null energy condition is violated quantum mechanically. Actually, the violation of the null energy condition is rather crucial for the consistency of quantum gravity in various ways. The Hawking radiation violates the classical area non-decreasing theorem proved by the null energy condition. The violation of the null energy condition in $(1+1)$-dimensional worldsheet makes graviton massless in string theory. Orientifolds as they are also break the null energy condition, but they are crucial ingredients in string dualities (see e.g. [7] for a study how objects with the negative tension will affect the second law in string theory).

To relax the null energy condition while still avoiding pathological space–time, there have been various modifications proposed. One promising direction is to average over the null geodesics. The so-called averaged null energy condition seems to hold in any quantum states in Minkowski space–time [8–10], but the violations were reported in curved backgrounds [11–13] (technically we have to restrict ourselves to the achronal average because in compact space–time the averaged null energy condition is easily violated due to Casimir energy: in any way, even the achronal version is violated). The violation is either induced by quantum states or anomalous contributions in the energy–momentum tensor induced by trace anomaly in curved space–time.

Do reported violations of the null energy condition invalidate the holographic c-theorem? Unlike the favored violation we mentioned, we do not want the violation to kill the holographic c-theorem because we believe that c-theorem is universally true in dual unitary relativistic quantum field theories. Most of the reported violations are not immediately threatening to us because the holographic renormalization group flow takes place in a boundary Poincaré invariant setup, and we do not consider time-dependent non-vacuum process. However, there are known possibilities of violating null energy condition in a vacuum setup from an anomalous contribution to the energy–momentum tensor in curved space–time due to the trace anomaly. It is a universal

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violation in any theory and it has been used to violate even the averaged null energy condition [11–13].

In this Letter, we would like to address the question and answer in a positive way. With a further thought, the violation of the (averaged) null energy condition are not important unless they appear in the self-consistent background that solves the quantum modified gravity equation. Indeed, there has been no reported violation of the averaged null energy condition in the self-consistent background [14,15,10]. We will see that a suitably modified holographic c-function with a peculiar log correction is still monotonically decreasing in so far as we add the counterterm that removes a ghost mode of gravity. This can be regarded as a consequence of the self-consistency.

The organization of the Letter is as follows. In Section 2, we review the holographic renormalization group and holographic c-theorem in $(1+3)$ dimensions with possible higher derivative corrections. In Section 3, we discuss the anomalous contribution to the energy–momentum tensor and possible violation of the null energy condition to see the fate of the holographic c-theorem. In Section 4, we summarize our findings and discuss possible future directions.

2. Holographic renormalization group and c-theorem

As a starting point of our discussion, let us consider the holographic renormalization group flow and the holographic c-theorem in $(1+3)$-dimensional Einstein gravity

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\,(R + L_{\text{matter}}). \tag{1}$$

Throughout the Letter, the Planck length is set to be one. In holographic renormalization group flow, we will consider the asymptotically AdS space–time whose metric is

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}\,dx^\mu\,dx^\nu, \tag{2}$$

where $\eta_{\mu\nu} = (-1,+1,+1)$ is the three-dimensional flat Minkowski space–time metric, and $A(r) \to A_{\text{UV}} r$ as $r \to +\infty$ and $A(r) \to A_{\text{IR}} r$ as $r \to -\infty$ for the flow between two dual conformal field theories.

The holographic c-function, denoted by $a(r)$ for a conventional reason, is defined by

$$a(r) \equiv \frac{\pi^{3/2}}{\Gamma(3/2)(A'(r))^2}, \tag{3}$$

where $A'(r) = \frac{dA(r)}{dr}$. At the fixed point $r \to \pm\infty$ it was interpreted as the universal term in the entanglement entropy of the dual conformal field theory. By using the Einstein equation, one can compute the change of the holographic c-function along the holographic renormalization group flow as

$$\begin{aligned} a'(r) &= -\frac{2\pi^{3/2}}{\Gamma(3/2)(A'(r))^3}A''(r) \\ &= -\frac{\pi^{3/2}}{\Gamma(3/2)(A'(r))^3}\left(T^t{}_t - T^r{}_r\right) \geqslant 0, \end{aligned} \tag{4}$$

where $T_{\mu\nu}$ is the matter energy–momentum tensor, and the last inequality is the claimed holographic c-theorem. To justify the inequality, we have assumed that the null energy condition is satisfied so that $T^r{}_r - T^t{}_t \geqslant 0$.

The null energy condition demands that for any null vector k_μ such that $k_\mu k^\mu = 0$, the energy–momentum tensor must satisfy the inequality $T_{\mu\nu}k^\mu k^\nu \geqslant 0$. In our example, we choose $k^\mu = (1, e^{-A(r)}, 0, 0)$. We will discuss the "normalization" of the null vector later when we discuss the averaged condition, but it is irrelevant here. The null energy condition leads to $-A''(r) \geqslant 0$ in the holographic renormalization group flow, and it has played a crucial role in establishing the holographic c-theorem.

In addition to the null energy condition, there was an implicit technical assumption $A'(r) \geqslant 0$ in (4). This can be derived from the fact that $A'(r \to \pm\infty) > 0$ and $A''(r) \leqslant 0$ from the null energy condition [4].

Before introducing higher derivative corrections, we have a couple of comments here. The first observation, which will be useful later, is that the metric for the holographic renormalization (2) is conformally flat and the Weyl tensor $W_{\mu\nu\rho\sigma}$ vanishes. This is tightly related to the fact that our holographic renormalization group flow preserves the boundary Poincaré invariance.

The second comment is that given the recent success in proving the weak version of the c-theorem in $(1+3)$-dimensional quantum field theories [6] (with the lack of a non-perturbative proof of the strong version [16,17]), one may be tempted to only require the weak version of the holographic c-theorem (note, however, we are dealing with the $(1+2)$-dimensional boundary, where things are less clear). This is closely related to the averaged null energy condition. The averaged null energy condition only demands that $\int_\gamma T_{\mu\nu}k^\mu k^\nu\,d\lambda \geqslant 0$ over any (achronal) null geodesics γ with the affine parameter λ (such that $\partial_\mu \lambda k^\mu = 1$). As mentioned in the introduction, the averaged null energy condition is more difficult to violate than point-wise null energy condition. In any way, in our holographic c-theorem, we needed a slightly different averaging: $\int_\gamma T_{\mu\nu}k^\mu k^\nu f(\lambda)\,d\lambda \geqslant 0$, where $\lambda = r$ and $f(\lambda) = \frac{1}{(A'(r))^3}$ in order to show $a_{\text{UV}} - a_{\text{IR}} \geqslant 0$. Although we will focus on the strong c-theorem in the following, it would be interesting to understand the physical origin of this averaging.

The final comment is on the relation between holographic c-theorem and the holographic equivalence of scale invariance and conformal invariance. For the holographic equivalence to work, we need to assume a strict version of the null energy condition that demands that the matter must be trivial when the null energy condition is identically saturated [18,19]. With this respect, we recall that the null energy condition is not enough to exclude the pathological situation where the matter has zero kinetic energy, and in order to guarantee the unitarity, it is not sufficient. Whenever the null energy condition is violated, the statement of the strict null energy condition is obscure. Fortunately, due to the symmetry of the problem, the anomalous violation we will discuss in the next section does not play an important role there.

Now let us introduce higher derivative corrections to the holographic c-theorem argument [4]. We generalize the Einstein–Hilbert action with various curvature squared terms:

$$\begin{aligned} S = \frac{1}{2}\int d^4x\,\sqrt{g}\bigg(&\frac{6}{L^2}\alpha + R \\ &+ L^2\big(\lambda_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R^2 \\ &+ \tilde{\lambda}\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\rho\sigma}R_{\gamma\delta}{}^{\rho\sigma}\big)\bigg). \end{aligned} \tag{5}$$

Although for completeness we have added the parity odd Hirzebruch–Pontryagin term $\tilde{\lambda}\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\rho\sigma}R_{\gamma\delta}{}^{\rho\sigma}$, it is topological, and does not affect the bulk holographic renormalization group flow. Similarly, a particular combination of the parity even term with $(\lambda_1, \lambda_2, \lambda_3) = (1, -4, 1)\lambda$ gives the Euler density: $(\text{Euler} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2)$, so it will not affect the bulk holographic renormalization group flow, either. Therefore, without losing generality, we can set $\lambda_1 = \tilde{\lambda} = 0$.