Do the Modified Uncertainty Principle and Polymer Quantization predict same physics?

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ABSTRACT

In this Letter we study the effects of the Modified Uncertainty Principle as proposed in Ali et al. (2009) [5] in simple quantum mechanical systems and study its thermodynamic properties. We have assumed that the quantum particles follow Maxwell–Boltzmann statistics with no spin. We compare our results with the results found in the GUP and polymer quantum mechanical frameworks. Interestingly we find that the corrected thermodynamic entities are exactly the same compared to the polymer results but the length scale considered has a theoretically different origin. Hence we express the need of further study for an investigation whether these two approaches are conceptually connected in the fundamental level.

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1. Introduction

The idea that the uncertainty principle could be affected by gravity was first given by Mead [1]. Later modified commutation relations between position and momenta commonly known as Generalized Uncertainty Principle (or GUP) were given by candidate theories of quantum gravity (String Theory, Doubly Special Relativity (or DSR) Theory and Black Hole Physics) with the prediction of a minimum measurable length [2,3]. Similar kind of commutation relation can also be found in the context of Polymer Quantization in terms of Polymer Mass Scale [4].

The authors in [5] proposed a MUP[1] which is consistent with DSR theory, String Theory and Black Hole Physics and which says

$$[x_i, x_j] = [p_i, p_j] = 0, \tag{1}$$

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right], \tag{2}$$

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left[1 - 2\alpha\langle p\rangle + 4\alpha^2\langle p^2\rangle\right]$$
$$\geqslant \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2\rangle}} + 4\alpha^2\right)\Delta p^2 + 4\alpha^2\langle p\rangle^2 - 2\alpha\sqrt{\langle p^2\rangle}\right], \tag{3}$$

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[1] From now we denote this as MUP to distinguish it from the Generalized Uncertainty Principle (GUP) introduced in [14]

where $\alpha = \frac{l_0 l_{pl}}{\hbar}$. Here l_{pl} is the Plank length ($\approx 10^{-35}$ m). It is normally assumed that the dimensionless parameter l_0 is of the order unity. If this is the case then the α dependent terms are only important at or near the Plank regime. But here we expect the existence of a new intermediate physical length scale of the order of $\alpha\hbar = l_0 l_{pl}$. We also note that this unobserved length scale cannot exceed the electroweak length scale [5] which implies $l_0 \leqslant 10^{17}$. These equations are approximately covariant under DSR transformations but not Lorentz covariant [3]. These equations also imply

$$\Delta x \geqslant (\Delta x)_{min} \approx l_0 l_{pl} \tag{4}$$

and

$$\Delta p \leqslant (\Delta p)_{max} \approx \frac{M_{pl}c}{l_0} \tag{5}$$

where M_{pl} is the Plank mass and c is the velocity of light in vacuum. It can be shown that Eq. (2) is satisfied by the following definitions $x_i = x_{oi}$ and $p_i = p_{oi}(1 - \alpha p_o + 2\alpha^2 p_o^2)$, where x_{oi}, p_{oj} satisfy $[x_{oi}, p_{oj}] = i\hbar\delta_{ij}$. Here we can interpret p_{oi} as the momentum at low energies having the standard representation in position space ($p_{oi} = -i\hbar\frac{\partial}{\partial x_{oi}}$) with $p_o^2 = \sum_{i=1}^3 p_{oi}p_{oi}$ and p_i as the momentum at high energies. We can also show that the p^2 term in the kinetic part of any Hamiltonian can be written as [5]

$$p^2 \implies p_o^2 - 2\alpha p_o^3 + \mathcal{O}(\alpha^2) + \cdots. \tag{6}$$

Here we assume that terms $\mathcal{O}(\alpha^2)$ are much smaller in magnitude in comparison to terms $\mathcal{O}(\alpha)$ as $\alpha = l_0 l_{pl}$. The effect of this proposed MUP is well studied recently for some well-known physical systems in [5–10].

In this Letter we study two simple quantum systems (particle in a box and harmonic oscillator) in the MUP framework and calculate the thermodynamic entities for each case. Later we compare our results with those as predicted by GUP [13] and Polymer Quantum Mechanics [12] and interestingly we found that the physical interpretation remains the same provided that the polymer length scale is theoretically very different in its origin.

2. Particle in a box

As we have mentioned earlier that we are going to study the MUP corrected quantum particle in a box problem (considering the quantum particle have no spin), so here we again mention the p^2 term in the Hamiltonian can be replaced by $p_0^2 - 2\alpha p_0^3 + \mathcal{O}(\alpha^2)$ where we have already designed p_o earlier and for our purpose we can use $p_{oi} \equiv -i\hbar \frac{\partial}{\partial x_i}$.

We can easily write the MUP corrected Schrödinger equation for the particle in a box (in one dimension) as

$$\gamma \frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial^2 \Psi}{\partial x^2} + k^2 \Psi = 0, \tag{7}$$

where $\gamma = 2i\alpha\hbar$ and $k^2 = \frac{2mE}{\hbar^2}$. It is clear from Eq. (7) that if $\alpha = \gamma = 0$ we get back the usual known equation with respect to some boundary condition. Here we are going to solve Eq. (7) in some perturbative sense. If $\gamma = 0$ we get $\Psi \sim \sin k_o x$. The subscript '$_o$' refers to the normal situation where we have not considered the MUP effect and $k_o = \sqrt{\frac{2mE_o}{\hbar^2}}$.

Now if we use the approximation $\Psi \sim \sin k_o x$ we can re-write Eq. (7) as

$$\frac{\partial^2 \Psi}{\partial x^2} - \gamma k_o^2 \frac{\partial \Psi}{\partial x} + k^2 \Psi = 0, \tag{8}$$

where $\lim_{\gamma \to 0} k = k_o$. We can now write the solution for Ψ as

$$\Psi \sim e^{i\alpha\hbar k_o^2 x} \sin\left(\sqrt{\alpha^2 \hbar^2 k_o^4 + k^2} x\right), \tag{9}$$

where we have already exploited the boundary condition $\Psi(x = 0) = 0$.

With another condition $\Psi(x = L) = 0$, where L is the length of the box, we get the quantization relation

$$\sqrt{\alpha^2 \hbar^2 k_o^4 + k^2} L = n\pi. \tag{10}$$

After some straightforward rearrangement we can finally write

$$E_n = E_{on} - 2\alpha^2 m E_{on}^2, \tag{11}$$

where $E_{on} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$ is the energy eigenvalue if $\alpha = 0$ (standard case).

This particular problem was studied earlier [13] where the generalized commutation relation was considered to be the one well discussed in [14]. There also we can find a correction is proportional to the square of the minimum length (l_0) with respect to the generalized commutation relation used. We can see

$$E_{nGUP} = \frac{n^2 \pi^2 \hbar^2}{2mL^2} + l_0^2 \frac{n^4 \pi^4 \hbar^2}{3mL^4}. \tag{12}$$

So now if we compare Eq. (12) with (11) we can see that the correction is proportional to the square of the minimum length and its coefficient is also same but there is a difference in sign in the prefactor. So the intention of the correction term is not the same in both the cases.

Now this particular problem is also studied in the realm of Polymer Quantization [12] and the approximate spectrum is found out to be

$$E_{nPoly} = \frac{n^2 \pi^2 \hbar^2}{2mL^2} - \lambda^2 \frac{n^4 \pi^4 \hbar^2}{24mL^4}, \tag{13}$$

where $\lambda = \frac{\mu_o}{L}$ and μ_o is considered as a constant but related to the polymer length scale. Now if we compare this result with Eq. (11) we see that these two corrections are similar (if we neglect the numerical factor of 24). Authors in [12] have argued that this coincidence is not surprising since polymer systems have similar modifications to GUP in their corresponding uncertainty relation [4]. But the MUP is having a term linear in Planck length in the commutation relation and still we can find corrections that are nearly or exactly similar to those as predicted by GUP [13] or Polymer Quantization [12]. The first order correction term is quadratic in length scale for all the cases but the energies are reduced both in the MUP and Polymer framework.

Now we are going to apply the MUP corrected energy spectrum to calculate the canonical partition function and other thermodynamic quantities for the ideal gas assuming the quantum particles follow Maxwell–Boltzmann statistics with no spin. So we first calculate the MUP corrected partition function and it can be expressed as

$$Z(\beta) \approx \sum_{n=1}^{\infty} \exp\left(\frac{-c\beta n^2}{L^2}\right) + \frac{d\beta\alpha^2}{L^4} \sum_{n=1}^{\infty} n^4 \exp\left(\frac{-c\beta n^2}{L^2}\right) + \mathcal{O}(\alpha^4), \tag{14}$$

where $c = \frac{\pi^2 \hbar^2}{2m}$, $d = \frac{\pi^4 \hbar^4}{2m}$ and $\beta = \frac{1}{K_B T}$. From now on we are going to work in the unit where $\hbar = l_{pl} = 1$. If we apply the Poisson summation formula [15] we can re-write the partition function as

$$Z(\beta) \approx \left(\frac{mL^2}{2\pi\beta}\right)^{\frac{1}{2}} + \alpha^2 \left(\frac{3m}{2\beta}\right) \left(\frac{mL^2}{2\pi\beta}\right)^{\frac{1}{2}} + \cdots$$
$$\approx \left(\frac{mL^2}{2\pi\beta}\right)^{\frac{1}{2}} \left[1 + \alpha^2 \left(\frac{3m}{2\beta}\right) + \cdots\right]. \tag{15}$$

The first term is exactly the partition function for one-dimensional ideal gas. Now as we are using the MUP in a perturbative sense so for our approximation to be valid we require $\alpha \ll \sqrt{\frac{2\beta}{3m}}$ as $\alpha = \frac{l_0 l_{pl}}{\hbar}$ and $l_{pl} = \hbar = 1$. For the case of indistinguishable particles we use the relation $F = -\frac{1}{\beta} \ln(\frac{Z^{\mathcal{N}}}{\mathcal{N}!})$ for the Helmholtz free energy and it comes out to be

$$F \approx \frac{-\mathcal{N}}{\beta} \left[1 + \ln\left(\frac{L}{\mathcal{N}}\right) + \frac{1}{2} \ln\left(\frac{m}{2\pi\beta}\right) + \ln\left(1 + \frac{3\alpha^2 m}{2\beta}\right) + \cdots\right]. \tag{16}$$

The chemical potential ($\mu = \frac{\partial F}{\partial \mathcal{N}}$) and the entropy ($S = k_B \beta^2 \frac{\partial F}{\partial \beta}$) can also be calculated

$$\mu = \frac{-1}{\beta} \left[\ln\left(\frac{L}{\mathcal{N}}\right) + \frac{1}{2} \ln\left(\frac{m}{2\pi\beta}\right) + \ln\left(1 + \frac{3\alpha^2 m}{2\beta}\right) + \cdots\right] \quad \text{and} \tag{17}$$

$$S = \mathcal{N} K_B \left[\frac{3}{2} + \ln\left(\frac{L}{\mathcal{N}}\right) + \frac{1}{2} \ln\left(\frac{m}{2\pi\beta}\right) + \frac{3\alpha^2 m}{2\beta} + \ln\left(1 + \frac{3\alpha^2 m}{2\beta}\right) + \cdots\right]. \tag{18}$$