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Qubits and oriented matroids in four time and four space dimensions

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ABSTRACT

We establish a connection between 4-rebits (real qubits) and the Nambu–Goto action with target 'spacetime' of four time and four space dimensions ($(4+4)$-dimensions). We motivate the subject with three observations. The first one is that a 4-rebit contains exactly the same number of degree of freedom as a complex 3-qubit and therefore 4-rebits are special in the sense of division algebras. Secondly, the $(4+4)$-dimensions can be splitted as $(4+4)=(3+1)+(1+3)$ and therefore they are connected with an ordinary $(1+3)$-spacetime and with changed signature $(3+1)$-spacetime. Finally, we show how geometric aspects of 4-rebits can be related to the chirotope concept of oriented matroid theory.

Recently, through the identification of the coordinates x^μ of a bosonic string, in target space of $(2+2)$-signature, with a 2×2 matrix x^{ab}, Duff [1] was able to discover new hidden discrete symmetries of the Nambu–Goto action [2,3]. It turns out that the key mathematical tool in this development is the Cayley hyperdeterminant $\mathrm{Det}(b)$ [4] of the hypermatrix $b_a{}^{bc}=\partial_a x^{bc}$. A striking result is that $\mathrm{Det}(b)$ can also be associated with the four electric charges and four magnetic charges of a STU black hole in four-dimensional string theory [5]. Even more surprising is the fact that $\mathrm{Det}(b)$ makes also its appearance in quantum information theory by identifying $b_a{}^{bc}$ with a complex 3-qubit system $a_a{}^{bc}$ [6]. These coincidences, among others, have increased the interest on the qubit/black hole correspondence [7].

It has been shown [8] that a straightforward generalization of the above Duff's formalism, concerning the Nambu–Goto action, can be applied to a target space of $(5+5)$-signature, but not to a space of $(4+4)$-signature. But, since in principle, the $(5+5)$-signature may be associated with a 5-qubit and the $(4+4)$-signature with a 4-qubit this is equivalent to say that the Nambu–Goto action exhibit discrete symmetries for a 5-qubit system, but not for a 4-qubit system.

On the other hand, in quantum information theory it does not seem to be any particular reason for avoiding unnormalized 4-qubits. In fact, a 4-qubit is just one possibility out of the complete set of N-qubit systems. It turns out that, in a particular subclass of N-qubit entanglement, the Hilbert space can be broken into the form $C^{2^N}=C^L\otimes C^l$, with $L=2^{N-1}$ and $l=2$. Such a partition it allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of 2-planes in C^L via the Plücker embedding. In this case, the Plücker coordinates of Grassmannians $Gr(L,l)$ are natural invariants of the theory. It turns out that in this scenario the complex 3-qubit, 4-qubit and 5-qubit admit a geometric interpretation in terms of the complex Grassmannians $Gr(4,2)$, $Gr(8,2)$ and $Gr(16,2)$, respectively (see Refs. [9] and [10] for details).

Of course, in this context, it has been mentioned in Ref. [11], and proved in Refs. [12] and [13], that for normalized qubits the complex 1-qubit, 2-qubit and 3-qubit are deeply related to division algebras via the Hopf maps, $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$, respectively. It seems that there does not exist a Hopf map for higher N-qubit states. So, from the perspective of Hopf maps, and therefore of division algebras, one arrives to the conclusion that 1-qubit, 2-qubit and 3-qubit are more special than higher-dimensional qubits (see Refs. [11–13] for details).

How can we make sense out of these different scenarios in connection with a 4-qubit system? Before we try to answer this question, let us think in a 3-qubit/black hole correspondence. In this case the symmetry of a extremal STU black hole model is $SL(2,R)^{\otimes 3}$. However in the case of a complex qubit system the symmetry group is $SL(2,C)^{\otimes 3}$. So, the problem is equivalent to an embedding of a real 3-qubit (3-rebit, see Ref. [14] for definition of N-rebits) relevant in STU black holes into complex 3-qubit in complex geometry. It has been shown [9] that this kind of embedding is not trivial and in fact requires the mathematical tools of fiber bundles with Grassmannian variety as a base space. It has been compared [10] this mechanism with the analogue situation

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described in twistor theory when one pass from real to complex Minkowski space (see also Refs. [15–17]).

Apart from these embeddings one may gain some insight on the above subject if one simple counts the number of degrees of freedom corresponding to the complex 3-qubit and 4-qubit and compare them with the corresponding real qubits, 3-rebit, 4-rebit. Consider the general complex state $|\psi\rangle \in C^{2^N}$,

$$|\psi\rangle = \sum_{a_1,a_2,\ldots,a_N=0}^{1} a_{a_1a_2\ldots a_N}|a_1a_2\cdots a_N\rangle, \tag{1}$$

where the states $|a_1a_2\cdots a_N\rangle = |a_1\rangle \otimes |a_2\rangle \cdots \otimes |a_N\rangle$ correspond to a standard basis of the N-qubit. For a 3-qubit (1) becomes

$$|\psi\rangle = \sum_{a_1,a_2,a_3=0}^{1} a_{a_1a_2a_3}|a_1a_2a_3\rangle, \tag{2}$$

while for 4-qubit one has

$$|\psi\rangle = \sum_{a_1,a_2,a_3,a_4=0}^{1} a_{a_1a_2a_3a_4}|a_1a_2a_3a_4\rangle. \tag{3}$$

One observes that $a_{a_1a_2a_3}$ has 8 complex degrees of freedom, that is 16 real degrees of freedom, while $a_{a_1a_2a_3a_4}$ contains 16 complex degrees of freedom, that is 32 real degrees of freedom. Let us denote N-rebit system (real N-qubit) by $b_{a_1a_2\ldots a_N}$. So we shall denote the corresponding 3-rebit, 4-rebit by $b_{a_1a_2a_3}$ and $b_{a_1a_2a_3a_4}$, respectively. One observes that $b_{a_1a_2a_3}$ has 8 real degrees of freedom, while $b_{a_1a_2a_3a_4}$ has 16 real degrees of freedom. Thus, by this simple (degree of freedom) counting one note that it seems more natural to associate the 4-rebit $b_{a_1a_2a_3a_4}$ with the complex 3-qubit, $a_{a_1a_2a_3}$, than with the complex 4-qubit, $a_{a_1a_2a_3a_4}$. Of course, by imposing some constraints one can always reduce the 32 real degrees of freedom of $a_{a_1a_2a_3a_4}$ to 16, and this is the kind of embedding discussed in Ref. [9]. Here, we shall focus in the first possibility, that is we associate the 4-rebit $b_{a_1a_2a_3a_4}$ with the 3-qubit $a_{a_1a_2a_3}$. The whole idea is to make sense out of a 4-rebit in the Nambu–Goto context without loosing the important connection with a division algebra via the Hopf map $S^{15} \xrightarrow{S^7} S^8$. Since from the point of view of division algebra the 3-qubit is special one may argue that 4-rebit is also special and therefore the $(4+4)$-signature must also be special. Motivated by this observation one may now proceed to recall why a straightforward application of Duff's prescription cannot be applied to the 4-rebit. The main purpose of this Letter is to propose a solution for a connection between 4-rebit and Nambu–Goto action.

Before we proceed further let us add other sources of motivation concerning the $(4+4)$-signature. First, we all agree that at macroscopic scales a general description of our world requires $(1+3)$-dimensions (a manifold of one time dimension and three space dimensions). But even for no experts it is evident the lack of symmetry between the number of time and space dimensions of our world. A natural question is: Why nature did not choose instead of $(1+3)$-dimensions other more symmetric combinations, such as $(1+1)$, $(2+2)$ or $(4+4)$-dimensions? Of course, one may expect that any complete unified theory must explain no only the number of dimensions of the spacetime but also its signature [18]. In the lack of such a unified theory it turns out convenient to explore separate signatures and dimensions. In this context it has been shown that the cases $(1+1)$ and $(2+2)$ may be considered as exceptional signatures [19]. We shall prove that in the context of the Nambu–Goto action the target space of $(4+4)$-dimensions can be understood as two copies of the $(2+2)$-dimensions. Roughly speaking, one may note that this is true because $(4+4) = ((2+2)+(2+2))$. Another similar motivation can be found if one considers the combination $(4+4) = ((3+1)+(1+3))$. In other words the $(4+4)$-dimensions can be splitted in the usual $(1+3)$-dimensions and in $(3+1)$-dimensions. It turns out that the case $(3+1)$-dimensions can be considered simply as a change of signature of $(1+3)$-dimensions [20]. So, $(4+4)$-dimensions must contains the usual $(1+3)$-dimensions of our world and a mirror $(3+1)$-dimensions with the signature changed.

Let us start by showing first that straightforward application of the Duff's formalism concerning the Nambu–Goto action/qubits correspondence works for $(2+2)$-signature, but no for the $(4+4)$-signature. For the case of $(2+2)$-signature, consider the identification,

$$\begin{aligned} &x^{11} \leftrightarrow x^1 + x^3, \qquad x^{12} \leftrightarrow x^2 + x^4, \\ &x^{21} \leftrightarrow x^2 - x^4, \qquad x^{22} \leftrightarrow -x^1 + x^3. \end{aligned} \tag{4}$$

Of course, this is equivalent to consider the matrix

$$x^{ab} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ x^2 - x^4 & -x^1 + x^3 \end{pmatrix}. \tag{5}$$

It is not difficult to prove that

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu}, \tag{6}$$

can also be written as

$$ds^2 = \frac{1}{2} dx^{ab} dx^{cd} \varepsilon_{ac}\varepsilon_{bd}, \tag{7}$$

where

$$\eta_{\mu\nu} = \mathrm{diag}(-1, -1, 1, 1), \tag{8}$$

is a flat metric corresponding to $(2+2)$-signature and ε_{ab} is the completely antisymmetric symbol (ε-symbol) with $\varepsilon_{12} = 1$. Note that (7) is invariant under $SL(2, R)^{\otimes 2}$ transformations.

We shall now show that a generalization of (6) and (7) to a target space of $(4+4)$-signature leads to a line element identically equal to zero. In this case the corresponding expressions similar to (4) are

$$\begin{aligned} &x^{111} \leftrightarrow x^1 + x^5, \qquad x^{121} \leftrightarrow x^2 + x^6, \\ &x^{211} \leftrightarrow x^2 - x^6, \qquad x^{221} \leftrightarrow -x^1 + x^5, \\ &x^{112} \leftrightarrow x^3 + x^7, \qquad x^{122} \leftrightarrow x^4 + x^8, \\ &x^{212} \leftrightarrow x^4 - x^8, \qquad x^{222} \leftrightarrow -x^3 + x^7. \end{aligned} \tag{9}$$

This is equivalent to consider two matrices

$$x^{ab1} = \begin{pmatrix} x^1 + x^5 & x^2 + x^6 \\ x^2 - x^6 & -x^1 + x^5 \end{pmatrix}, \tag{10}$$

and

$$x^{ab2} = \begin{pmatrix} x^3 + x^7 & x^4 + x^8 \\ x^4 - x^8 & -x^3 + x^7 \end{pmatrix}. \tag{11}$$

At first sight one may consider the line element

$$ds^2 = \frac{1}{2} dx^{abc} dx^{def} \varepsilon_{ad}\varepsilon_{be}\varepsilon_{cf} \tag{12}$$

as the analogue of (7). But this vanishes identically because $s^{cf} \equiv dx^{abc}dx^{def}\varepsilon_{ad}\varepsilon_{be}$ is a symmetric quantity, while ε_{cf} is antisymmetric.

Similarly, it is not difficult to show [1] (see also Ref. [8]) that the world sheet metric in $(2+2)$-dimensions