Physics Letters B 718 (2013) 1552–1556

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Spherical null geodesics of rotating Kerr black holes

Shahar Hod [a,b,*]

[a] The Ruppin Academic Center, Emeq Hefer 40250, Israel
[b] The Hadassah Institute, Jerusalem 91010, Israel

ARTICLE INFO

Article history:
Received 19 November 2012
Accepted 19 December 2012
Available online 21 December 2012
Editor: A. Ringwald

ABSTRACT

The non-equatorial spherical null geodesics of rotating Kerr black holes are studied *analytically*. Unlike the extensively studied equatorial circular orbits whose radii are known analytically, no closed-form formula exists in the literature for the radii of generic (non-equatorial) spherical geodesics. We provide here an approximate formula for the radii $r_{\rm ph}(a/M;\cos i)$ of these spherical null geodesics, where a/M is the dimensionless angular momentum of the black hole and $\cos i$ is an effective inclination angle (with respect to the black-hole equatorial plane) of the orbit. It is well-known that the equatorial circular geodesics of the Kerr spacetime (the prograde and the retrograde orbits with $\cos i=\pm 1$) are characterized by a *monotonic* dependence of their radii $r_{\rm ph}(a/M;\cos i=\pm 1)$ on the dimensionless spin-parameter a/M of the black hole. We use here our novel analytical formula to reveal that this well-known property of the equatorial circular geodesics is actually *not* a generic property of the Kerr spacetime. In particular, we find that counter-rotating spherical null orbits in the range $(3\sqrt{3}-\sqrt{59})/4 \lesssim \cos i<0$ are characterized by a *non*-monotonic dependence of $r_{\rm ph}(a/M;\cos i=\text{const})$ on the dimensionless rotation-parameter a/M of the black hole. Furthermore, it is shown that spherical photon orbits of rapidly-rotating black holes are characterized by a critical inclination angle, $\cos i=\sqrt{4/7}$, above which the coordinate radii of the orbits approach the black-hole radius in the extremal limit. We prove that this critical inclination angle signals a transition in the physical properties of the spherical null geodesics: in particular, it separates orbits which are characterized by finite proper distances to the black-hole horizon from orbits which are characterized by infinite proper distances to the horizon.

1. Introduction

The characteristic geodesics of the Kerr black-hole spacetime have been extensively studied since the pioneering work of Carter [1], see also [2–7] and references therein. Of particular importance are spherical null orbits—orbits with constant coordinate radii on which massless particles can orbit the black hole. The spherical null geodesic ("photonsphere") of a black hole provides valuable information on the structure and geometry of the black-hole spacetime.

The spherical null orbits are especially interesting from both an astrophysical and theoretical points of view. For example, the optical appearance to external observers of a star undergoing gravitational collapse is related to the physical properties of the photonsphere [6,8,9]. This surface also determines how the night sky would appear to an observer near a black hole or a very compact star [10]. In addition, spherical null orbits are closely related to the characteristic scattering resonances of black holes in the eikonal limit (the geometric-optics approximation) [6,11–16]. According to the geometric-optics approximation, these characteristic black-hole quasinormal resonances correspond to massless particles trapped at the unstable null orbit and slowly leaking out [6,11–16].

Furthermore, it was recently proved that for hairy black-hole configurations, the black-hole photonsphere provides a generic lower bound on the effective length of the hair [17,18]. In addition, it was recently proved [19] that circular null geodesics provide the fastest way to circle a black hole as measured by asymptotic observers.

It is worth emphasizing that in most cases of physical interest [20,21] the physical properties of the spherical null geodesics (and in particular, the radii and the energy-to-angular-momentum ratio) must be computed *numerically*. The only known exceptions are the prograde and retrograde circular orbits in the equatorial plane of the rotating Kerr black hole, in which case closed analytical formulae for the physical properties of the orbits have been provided in [2], see Eq. (21) below. The parameters of the null polar orbit are also known in a closed form [5], see Eq. (22) below.

To the best of our knowledge, no closed-form formula exists in the literature for the radii of generic (*non*-equatorial) spherical null geodesics of rotating Kerr black holes. One of the main goals of

* Correspondence to: The Ruppin Academic Center, Emeq Hefer 40250, Israel.
E-mail address: shaharhod@gmail.com.

0370-2693/$ – see front matter
http://dx.doi.org/10.1016/j.physletb.2012.12.047

the present Letter is to derive effective *analytical* formulae which describe the physical properties of such non-equatorial spherical geodesics in the Kerr spacetime.

It is well-known that the equatorial (prograde and retrograde) circular orbits of the Kerr spacetime are characterized by a *monotonic* dependence of their radii $r_{\rm ph}(a)$ on the spin-parameter a of the black hole [2]. In the present study we shall use our new analytical formula (see Eq. (24) below) to show that this well-known property of the equatorial circular geodesics is actually *not* a generic property of the Kerr geometry. In particular, we shall show that there is a significant fraction of counter-rotating spherical orbits for which the function $r_{\rm ph}(a)$ exhibits a *non*-monotonic behavior.

Recently, Yang et al. [16] have studied the null spherical orbits of the Kerr spacetime. One of the most remarkable conclusions of [16] is that near-extremal Kerr black holes are characterized by a significant fraction of spherical null geodesics whose radii approach the black-hole radius in the near-extremal limit. In particular, it was observed numerically in [16] that such near-horizon orbits exist for near-extremal black holes in the finite interval

$$\sin\theta_c \equiv 0.731 \lesssim \sin\theta \leqslant 1, \tag{1}$$

where θ is the inclination angle of the orbit with respect to the polar axis. Below we shall provide a fully *analytical* explanation for this phenomena. Furthermore, we shall obtain an analytical expression for the exact value of the critical polar angle $\sin\theta_c$ above which the near-horizon spherical null geodesics appear.

The rest of the Letter is devoted to the investigation of the physical properties of non-equatorial spherical null geodesics in the rotating Kerr spacetime. In Section 2 we describe the dynamical equations which determine the null geodesics of the Kerr spacetime. In Section 3 we obtain the characteristic equation which determines the radii of the spherical null geodesics. In Section 4 we solve the characteristic equation using a series expansion of $r_{\rm ph}(a/M;\cos i)$ in powers of the dimensionless rotation-parameter a/M of the black hole. We then use this analytic approach to reveal the regime in which the function $r_{\rm ph}(a/M)$ exhibits a non-monotonic behavior. In Section 5 we analyze the spherical null geodesics of rapidly-rotating black holes and discuss the near-horizon orbits of these near-extremal black holes. We conclude in Section 6 with a summary of the main results.

2. Description of the system

We shall analyze the spherical null geodesics which characterize the Kerr black-hole spacetime. In Boyer–Lindquist coordinates the metric is given by (we use gravitational units in which $G=c=1$) [3,22]

$$ds^2=-\Big(1-\frac{2Mr}{\rho^2}\Big)dt^2-\frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\phi+\frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2+\Big(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\rho^2}\Big)\sin^2\theta\,d\phi^2, \tag{2}$$

where M and a are the mass and angular momentum per unit mass of the black hole, respectively. Here $\Delta\equiv r^2-2Mr+a^2$ and $\rho\equiv r^2+a^2\cos^2\theta$. The black-hole (event and inner) horizons are located at the zeroes of Δ:

$$r_\pm = M\pm\big(M^2-a^2\big)^{1/2}. \tag{3}$$

Null geodesics in the rotating Kerr spacetime are characterized by three constants of the motion [1]. In terms of the covariant Boyer–Lindquist components of the 4-momentum, these conserved quantities are [2]:

$$E\equiv -p_t = \text{total energy}, \tag{4}$$

$$L_z\equiv p_\phi = \text{component of angular momentum parallel to the symmetry axis}, \tag{5}$$

and

$$Q\equiv p_\theta^2+\cos^2\theta\big[-a^2p_t^2+p_\phi^2/\sin^2\theta\big]. \tag{6}$$

The geodesics in the black-hole spacetime are governed by the following set of equations [2]

$$\rho\frac{dr}{d\lambda}=\pm\sqrt{V_r}, \tag{7}$$

$$\rho\frac{d\theta}{d\lambda}=\pm\sqrt{V_\theta}, \tag{8}$$

$$\rho\frac{d\phi}{d\lambda}=\big(L_z/\sin^2\theta-aE\big)+aT/\Delta, \tag{9}$$

$$\rho\frac{dt}{d\lambda}=a\big(L_z-aE\sin^2\theta\big)+\big(r^2+a^2\big)T/\Delta, \tag{10}$$

where λ is an affine parameter along the null geodesics. Here [2]

$$T\equiv E\big(r^2+a^2\big)-L_za, \tag{11}$$

$$V_r\equiv T^2-\Delta\big[(L_z-aE)^2+Q\big], \tag{12}$$

$$V_\theta\equiv Q-\cos^2\theta\big[-a^2E^2+L_z^2/\sin^2\theta\big]. \tag{13}$$

The effective potentials V_r and V_θ determine the orbital motions in the r and θ directions, respectively.

Equatorial orbits are characterized by $Q=0$ [2]. It is convenient to quantify the deviation of a generic (non-equatorial) orbit from the equatorial plane of the black hole using an effective inclination angle i which is defined by [23–26]

$$\cos i\equiv\frac{L_z}{L}, \tag{14}$$

where

$$L\equiv\sqrt{L_z^2+Q}\,. \tag{15}$$

Note that L and i are constants of the motion. For spherical black-hole spacetimes (with $a=0$), L is the total angular momentum of the particle [2]. The extensively studied simple equatorial geodesics are characterized by $\cos^2 i=1$, where $\cos i=+1/-1$ correspond to prograde/retrograde orbits, respectively. Polar orbits are characterized by $\cos i=0$.

3. Spherical null orbits in the black-hole spacetime

Spherical geodesics in the black-hole spacetime are characterized by the two conditions [2,3]

$$V_r=0\quad\text{and}\quad V_r'=0. \tag{16}$$

Substituting (12) into (16), one finds the ratio

$$\frac{L^2}{E^2}=\frac{3r^4+a^2r^2}{r^2-a^2\sin^2 i} \tag{17}$$

and the characteristic equation

$$r^5-3Mr^4+2a^2r^3\sin^2 i-2Ma^2r^2+a^4r\sin^2 i+Ma^4\sin^2 i + 2Mar\cos i\sqrt{3r^4+\big(1-3\sin^2 i\big)a^2r^2-a^4\sin^2 i}=0, \tag{18}$$

which determines the radii $r_{\rm ph}(a/M;\cos i)$ of the null spherical geodesics. It is convenient to define the dimensionless radial coordinate $\bar r$ and the dimensionless spin-parameter $\bar a$ of the black hole: