Physics Letters B 708 (2012) 309–313

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

More exact tunneling solutions in scalar field theory

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ARTICLE INFO

Article history:
Received 20 October 2011
Received in revised form 15 December 2011
Accepted 10 January 2012
Available online 12 January 2012
Editor: A. Ringwald

ABSTRACT

We present exact bounce solutions and amplitudes for tunneling in (i) a piecewise linear–quartic potential and (ii) a piecewise quartic–quartic potential, ignoring the effects of gravitation. We cross check their correctness by comparing with results obtained through the thin-wall approximation and with a piecewise linear–linear potential. We briefly comment on applications in cosmology.

1. Introduction

In recent times, first order phase transitions have gained significant interest, for example as sources of gravitational waves [1] and in transversing the string theory landscape [2,3]. In the latter picture, the scalar field potential possesses a plethora of local minima. A field that is initially trapped in a higher energy vacuum jumps to a lower energy vacuum via a quantum tunneling process.

The underlying microphysics of tunneling can be described by instantons, i.e. classical solutions of the Euclidean equations of motion of the system [4,5]. Tunneling proceeds via the nucleation of bubbles of true (or rather lower energy) vacuum surrounded by the sea of false vacuum. If the curvature of the potential is large compared to the corresponding Hubble scale, this process can be described by Coleman instantons, i.e. bounce solutions to the Euclidean equations of motion [4,5]. For relatively flat potentials, tunneling proceeds via Hawking–Moss instantons [6].

Ignoring the effects of gravity, Coleman presented a straightforward prescription for computing vacuum transitions [4]. The tunneling amplitude for a transition from the false (or higher energy) vacuum at ϕ_+ to the true (or lower energy) vacuum at ϕ_- is given by $A\exp(-B)$. The coefficient A is typically ignored but in principle calculable, see [7]. The exponent $B = S_E(\phi_B) - S_E(\phi_+)$ (sometimes also referred to as the bounce action) is the difference between the Euclidean action $S(\phi) = 2\pi^2 \int_0^\infty dr\, r^2(\frac{1}{2}\phi'^2 + V(\phi))$ for the spherically symmetric bounce solution ϕ_B and for the false vacuum ϕ_+. The bounce obeys the one-dimensional Euclidean equation of motion

$$\phi_B'' + \frac{3}{r}\phi_B' - \partial_\phi V(\phi_B) = 0, \tag{1}$$

where $\phi' \equiv \partial_r \phi$ and $r = \sqrt{t^2 - \vec{x}^2}$ is the radial coordinate of the spherical bubble. This configuration describes the bubble at the time of nucleation. In this Letter, we ignore its subsequent evolution, and focus on the computation of B.

In general, the Coleman bounce solutions can be computed exactly only for very few potentials. However, if the potential difference between the two vacua is small compared to the typical potential scale, the tunneling amplitude can be computed using the thin-wall approximation. Otherwise, one needs to resort to either numerical computations (see [8] for an approach for a generic quartic potential) or approximate the potential by potentials for which the exact instanton solutions are known. To the best of our knowledge, only for very few potentials has the Coleman tunneling process been solved analytically: a piecewise linear–linear potential [9] and piecewise linear–quadratic potentials [10–12]. While the Letter was being finished, we became aware of [13] who presented a bounce solution for tunneling in a quartic–linear potential. A different approach was taken by [14] who reconstruct fully analytically tractable potentials, including the effects of gravity, from analytically exact bubble geometries.

We present new exact solutions for tunneling within piecewise potentials where the true vacuum potential is a quartic, see Figs. 1 and 2. The potential for $\phi > 0$ ("on the right") is given by

$$V_R(\phi) = V_T - \Delta V_- + \frac{\Delta V_-}{\phi_-^4}(\phi - \phi_-)^4, \tag{2}$$

where $\Delta V_- \equiv V_T - V_-$. For simplicity, we chose $\phi = 0$ as the matching point and $V(\phi = 0) = V_T$. We will choose the potential for $\phi < 0$ ("on the left") as either linear or quartic and discuss the solutions in Sections 2 and 3 respectively.

For each piecewise potential, we proceed analogously to [9,12]: First we solve the equation of motion for the scalar field in $V_R(\phi)$, subject to the boundary condition at the center of the bubble $\phi_R(0) = \phi_0$, $\phi_R'(0) = 0$. We assume that the bubble nucleation point is located at $\phi_0 > 0$, i.e. it is in the valley of the true vacuum.

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0370-2693/$ – see front matter
doi:10.1016/j.physletb.2012.01.026

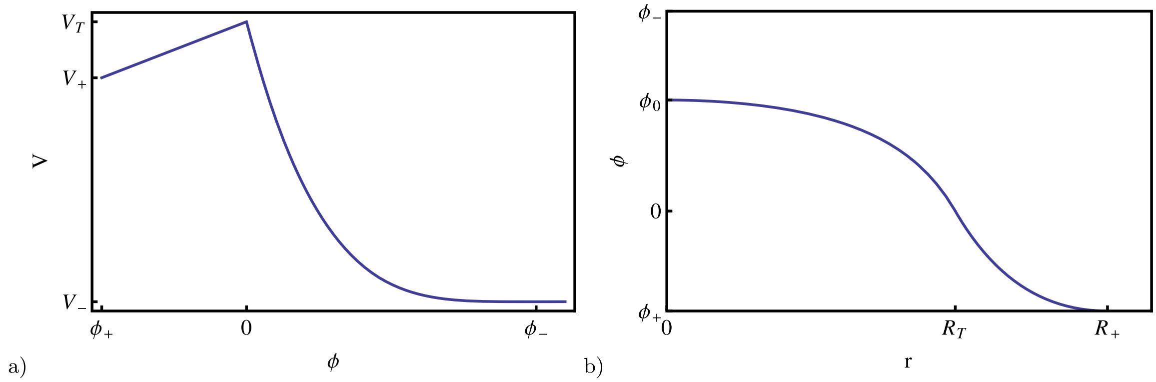

Fig. 1. (a) Schematic plot of the piecewise linear–quartic potential. The left part of the potential is a linear function of ϕ, the right part – a quartic function. The bounce describes tunneling from the field sitting in the false vacuum at ϕ_+ towards the true vacuum located at ϕ_-. (b) Schematic view of the bounce solution for (a). Inside the bubble at $r=0$, the field is at $\phi_0>0$. The bubble wall is located around R_T, but not necessarily thin. Outside of the bubble at $r=R_+$, the field is still in the false vacuum.

Then, we solve the equation of motion for the field in V_L, subject to $\phi_L(R_+)=\phi_+$, $\phi_L'(R_+)=0$. In other words, we assume that at some radius R_+ (which can be ∞) outside of the bubble of true vacuum, the field sits in the false vacuum. Then, we match the solutions at some radius R_T by enforcing $\phi_L(R_T)=\phi_R(R_T)=0$ and $\phi_L'(R_T)=\phi_R'(R_T)$. This allows us to determine the constants R_T, R_+, and ϕ_0. Here, R_T is roughly the radius of the bubble when it materializes at $\phi=\phi_0$, whereas the value comparing R_+ to R_T gives us an idea about the width of the bubble wall.

It is then straightforward to integrate the action for ϕ_L and ϕ_R, obtaining B. We compare the tunneling bounce action B for the piecewise linear–quartic potential with the results of both the thin-wall approximation and the piecewise linear–linear potential solved in [9]. Finally, we compute the tunneling amplitude for the piecewise quartic–quartic potential and compare it with the results obtained using the thin-wall approximation, as well as with the tunneling amplitude in a piecewise linear–quartic potential.

2. Linear on the left, quartic on the right

In this section we compute the tunneling rate for a piecewise potential of the form

$$V(\phi)=\begin{cases} V_T-\frac{\Delta V_+}{\phi_+}\phi, & \phi\leqslant 0,\\ V_T-\Delta V_-+\frac{\Delta V_-}{\phi_-^4}(\phi-\phi_-)^4, & \phi>0,\end{cases} \tag{3}$$

where $\Delta V_-\equiv V_T-V_-=\frac{\lambda_4}{4}\phi_-^4$ and $\Delta V_+\equiv V_T-V_+=-\lambda_1\phi_+$ are the depths of the true and false minimum, see Fig. 1. Subject to the boundary conditions $\phi_R(0)=\phi_0$, $\phi_R'(0)=0$, solving the equation of motion of the bounce, i.e. Eq. (1) on the right side of the potential, we have [15]

$$\phi_R(r)=\phi_-+\frac{2(\phi_0-\phi_-)}{2-\frac{\Delta V_-(\phi_0-\phi_-)^2}{\phi_-^4}r^2}. \tag{4}$$

Similarly on the left side of the potential, subject to $\phi_L(R_+)=\phi_+$, $\phi_L'(R_+)=0$, we have the bounce solution

$$\phi_L(r)=\phi_+-\frac{\Delta V_+}{8\phi_+}\frac{(r^2-R_+^2)^2}{r^2}. \tag{5}$$

A schematic view of the bounce is shown in Fig. 1(b).

We now determine the constants R_+ and ϕ_0 by solving the matching equations for the two solutions $\phi_R(R_T)=0$, $\phi_L(R_T)=0$. Using the first condition, we get ϕ_0 in terms of R_T

$$\phi_0=\frac{\phi_-^3}{\Delta V_-R_T^2}\left[\frac{\Delta V_-R_T^2}{\phi_-^2}+\left(1-\sqrt{\frac{2\Delta V_-R_T^2}{\phi_-^2}+1}\right)\right], \tag{6}$$

while the second condition gives

$$R_+=\sqrt{R_T\left(R_T+\frac{2\sqrt{2}\alpha\phi_-}{\sqrt{\Delta\Delta V_-}}\right)}. \tag{7}$$

Here, we have introduced $\Delta=\Delta V_+/\Delta V_-$ and $\alpha=-\phi_+/\phi_-$. Similarly, using the smoothness of the solution at R_T, i.e. $\phi_R'(R_T)=\phi_L'(R_T)$, we find

$$R_T=\frac{\phi_-(\sqrt{\Delta}(1+2\alpha)+\sqrt{4\alpha(1+\alpha)+\Delta})}{(1-\Delta)\sqrt{2\Delta V_-}}. \tag{8}$$

Computing the exponent of the tunneling amplitude in terms of R_T gives

$$B=\frac{\pi^2}{6\Delta V_-}\Bigg\{3R_T^4(\Delta-1)\Delta V_-^2+8\sqrt{2}R_T^3\alpha\Delta V_-\sqrt{\Delta\Delta V_-}\phi_- +2\phi_-^4\left[-1+\sqrt{1+\frac{2R_T^2\Delta V_-}{\phi_-^2}}\right] +2R_T^2\Delta V_-\phi_-^2\left[\left(6\alpha^2-3\right)+2\sqrt{1+\frac{2R_T^2\Delta V_-}{\phi_-^2}}\right]\Bigg\}. \tag{9}$$

Plugging R_T from Eq. (8), we obtain a rather monstrous expression

$$B=\frac{\pi^2\phi_-^4}{6\Delta V_-}\Bigg\{4\alpha\sqrt{\Delta}\left[\frac{(1+2\alpha)\sqrt{\Delta}+\sqrt{4\alpha(1+\alpha)+\Delta}}{1-\Delta}\right]^3 -\frac{3}{4}\left[\frac{(1+2\alpha)\sqrt{\Delta}+\sqrt{4\alpha(1+\alpha)+\Delta}}{(1-\Delta)^{3/4}}\right]^4 +\left[\frac{(1+2\alpha)\sqrt{\Delta}+\sqrt{4\alpha(1+\alpha)+\Delta}}{1-\Delta}\right]^2\Bigg[-3+6\alpha^2 +2\sqrt{1+\left[\frac{(1+2\alpha)\sqrt{\Delta}+\sqrt{4\alpha(1+\alpha)+\Delta}}{1-\Delta}\right]^2}\Bigg] +2\Bigg[-1+\sqrt{1+\left[\frac{(1+2\alpha)\sqrt{\Delta}+\sqrt{4\alpha(1+\alpha)+\Delta}}{1-\Delta}\right]^2}\Bigg]\Bigg\}. \tag{10}$$

To cross check our result, we take the thin-wall limit of Eq. (10) by replacing $\Delta=1-\frac{\epsilon}{\Delta V_-}$, where ϵ is the energy difference between the true and false vacua. In the thin-wall limit $\epsilon\ll V_T$. Performing a series expansion around $\epsilon=0$, the lowest order term