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Area law of the entropy in the critical gravity

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ABSTRACT

The entropy of the Schwarzschild-anti de Sitter black hole in the recently proposed four-dimensional critical gravity is trivial in the Euclidean action formulation, while it can be expressed by the area law in terms of the brick wall method given by 't Hooft. To resolve this discrepancy, we relate the Euclidean action formulation to the brick wall method semiclassically, and show that the entropy of the black hole can be expressed by the area law even at the critical point.

1. Introduction

Since it has been claimed that the general relativity is non-renormalizable, there have been extensive studies for quantum theory of gravity such as string theory, conventional perturbative gravity, and so on. In particular, a perturbatively renormalizable gravity theory can be built by adding quadratic curvature terms to the Einstein gravity [1,2]. However, theories including higher-order time-derivative terms should endure massive ghost modes. In recent studies on the three-dimensional topologically massive gravity [3,4] including a cosmological constant, it has been shown that there exists some critical point such that the massive mode becomes massless and carries no energy, so that the problem can be solved [5]. Similarly, in the four-dimensional quadratic gravity theory with a cosmological constant, one can find a critical point, where the massive ghost mode disappears.

The model called the *critical gravity* [6] can be defined by

$$I_{\mathrm{CG}}[g]=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda+\alpha\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)\right], \tag{1}$$

where Λ is a cosmological constant. At the critical point, $\alpha=3/2\Lambda$, in spite of the renormalizability, it seems to be trivial in the sense that the mass and the entropy of a Schwarzschild-anti de Sitter (SAdS) black hole which is a solution to this theory become zeroes [6]. Moreover, these results can be also confirmed by the Euclidean action formulation of the black hole thermodynamics [7–9].

On the other hand, it has been suggested by Bekenstein that the intrinsic entropy of a black hole is proportional to the surface area at the event horizon [10–12], and then the quantum field theoretic calculation for the Schwarzschild black hole has been given by Hawking [13]. Actually, one of the best way to reproduce the area law of black holes is to use the brick wall method suggested by 't Hooft [14]. By considering the fluctuation of a matter field around the black hole semiclassically, one can always get the desired results; however, this result is not compatible with the result of the Euclidean action formulation for the critical gravity.

In this Letter, we would like to resolve the above-mentioned issue and study how to derive the entropy satisfying the area law in the Euclidean action formulation. First task is to get a nontrivial free energy by taking into account higher-order loop corrections in the Euclidean path integral and then the corresponding entropy may be nontrivial. For convenience, the fluctuation of the metric field will be ignored, i.e. our calculations will be performed in semiclassical approximations. We recapitulate the Euclidean action formulation by carefully considering the appropriate boundary term in Section 2. In contrast to conventional cases, the entropy is trivially zero, assuming the critical condition. It means that the partition function is trivial so that the area law of the entropy does not appear. So, we consider the one loop correction of the scalar

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degrees of freedom around the black hole in Section 3 and relate the Euclidean action formulation to the brick wall method semiclassically. Eventually, the free energy turns out to be nontrivial even at the critical condition. It is actually compatible with the free energy obtained from the brick wall method along with some conditions. As a result, it gives the area law of the black hole entropy in Section 4. Finally, in Section 5, summary and some discussions are given.

2. Thermodynamics with the Euclidean action formulation

We start with a scalar field ϕ minimally coupled to the critical gravity as $I_{\rm tot} = I_{\rm CG} + I_\phi$, where the scalar field action is

$$I_\phi[g,\phi] = -\int d^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]. \tag{2}$$

For $\phi = 0$, the SAdS black hole is just a classical solution to this model. The line element of the SAdS black hole is given by $ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega_2^2$ with

$$\begin{aligned} f(r) &= 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2 \\ &= \left(1 - \frac{r_h}{r}\right)\left[1 - \frac{\Lambda}{3}\left(r^2 + r_h r + r_h^2\right)\right], \end{aligned} \tag{3}$$

where $M = (r_h/2G)(1 - \Lambda r_h^2/3) > 0$ is the mass parameter of the black hole, $\Lambda < 0$ is the cosmological constant, and r_h is the radius of the horizon. The free energy $F^{(0)}$ for this vanishing scalar solution can be obtained from the Euclidean action formulation [7–9],

$$Z^{(0)}[g] = \exp\bigl(iI_{\rm CG}[g]\bigr) = \exp\bigl(-\beta F^{(0)}\bigr). \tag{4}$$

The crucial ingredient for this calculation is to find the consistent boundary term. Following Ref. [15], an auxiliary field $f_{\mu\nu}$ is introduced to localize the higher curvature terms so that the Euclidean version of the action (1) and the corresponding boundary term can be written in the form of

$$\begin{aligned} I_{\rm CG} = &-\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\biggl[R - 2\Lambda + f^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \\ &- \frac{1}{4\alpha}f^{\mu\nu}(f_{\mu\nu} - g_{\mu\nu}f)\biggr], \end{aligned} \tag{5}$$

$$I_{\rm B} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}\bigl[2K + \hat{f}^{ij}(K_{ij} - \gamma_{ij}K)\bigr], \tag{6}$$

where γ_{ij} and K_{ij} are the induced metric and the extrinsic curvature of the boundary, respectively. And $\hat{f}^{ij}$ in the boundary term is defined as $\hat{f}^{ij} = f^{ij} + f^{ri}N^j + f^{rj}N^i + f^{rr}N^iN^j$ with $N^i = -g^{ri}/g^{rr}$ for the hypersurface described by $r = r_0$. In the Euclidean geometry, the Euclidean time is defined by $\tau = it$ and should be identified by $\tau = \tau + \beta_H$ to avoid a conical singularity at the event horizon, where β_H is the inverse of the Hawking temperature.

Next, taking the boundary to the infinity $r_0 \to \infty$, the free energy is obtained as

$$F^{(0)} = \beta_H^{-1}(I - I_{\rm vacuum}) = [1 - 2\alpha\Lambda/3]\frac{r_h}{4G}\left(1 + \frac{\Lambda}{3}r_h^2\right), \tag{7}$$

where $I = I_{\rm CG} + I_{\rm B}$ and $I_{\rm vacuum} = I|_{M=0}$. The Hawking temperature for the given metric function (3) is calculated as

$$T_H = \beta_H^{-1} = \frac{1 - \Lambda r_h^2}{4\pi r_h}. \tag{8}$$

Then, the thermodynamic first law reads the entropy and the energy of the black hole,

$$S^{(0)} = \beta_H^2\frac{\partial F^{(0)}}{\partial\beta_H} = [1 - 2\alpha\Lambda/3]\frac{\pi r_h^2}{G}, \tag{9}$$

$$E^{(0)} = F^{(0)} + \beta_H^{-1}S^{(0)} = [1 - 2\alpha\Lambda/3]\frac{r_h}{2G}\left(1 - \frac{\Lambda}{3}r_h^2\right), \tag{10}$$

which are exactly same with those obtained in Ref. [6]. Note that the factor $[1 - 2\alpha\Lambda/3]$ vanishes at the critical point $\alpha = 3/2\Lambda$. Thus, we can confirm that the energy and the entropy of the SAdS black hole at the critical point vanish in the Euclidean action formulation.

As was mentioned in the previous section, the entropy from the brick wall method satisfies the area law, which is connected with the nontrivial thermodynamic quantities such as energy and heat capacity. At first glance, there seems to exist incompatibility between the Euclidean action formulation and the brick wall method. In what follows, we shall show that the semiclassical treatment of the Euclidean action formulation can be related to the brick wall method.

3. Semiclassical Euclidean action formulation

Now, we take the classical background as the SAdS black hole metric along with $\phi = 0$, and then consider the fluctuated quantum field semiclassically. The partition function up to one loop order for the scalar field is expressed as

$$\begin{aligned} Z[g] &= Z^{(0)}[g]Z^{(1)}[g] \\ &= \exp\bigl(-\beta F^{(0)}\bigr)\exp\bigl(-\beta F^{(1)}\bigr) \\ &= e^{iI_{\rm CG}[g]}\int\mathcal{D}\phi e^{iI_\phi[g,\phi]}, \end{aligned} \tag{11}$$

where the total free energy consists of $F = F^{(0)} + F^{(1)}$. Note that the tree level free energy $F^{(0)}$ is trivial at the critical point as seen in the previous section, so that the nontrivial contribution to the free energy should come from the one loop effective action.

The one loop partition function $Z^{(1)}$ can be written as

$$\begin{aligned} Z^{(1)}[g] &= \int\mathcal{D}\phi e^{iI_\phi} \\ &= \det{}^{-1/2}\bigl(-\Box + m^2\bigr), \end{aligned} \tag{12}$$

and the effective action W_ϕ becomes

$$\begin{aligned} W_\phi &= \frac{i}{2}\ln\det\bigl(-\Box + m^2\bigr) \\ &= \frac{i}{2}\operatorname{Tr}\ln\bigl(-\Box + m^2\bigr) \\ &= \frac{i}{2}\int\frac{d^4x\,d^4k}{(2\pi)^4}\ln\bigl(k_\mu k^\mu + m^2\bigr), \end{aligned} \tag{13}$$

where k_μ is the conjugate momentum of x^μ. Note that a (covariant) Fourier transform in curved spacetimes has not been established [16–19]. However, the manifold can be split into a number of small pieces, in which we can consider a Riemann normal coordinates, i.e. $\int_{\mathcal{M}} d^4x\sqrt{-g} \simeq \sum_{U\subset\mathcal{M}}\int_U d^4\tilde{x}$, where $\tilde{x}$ represents the Riemann normal coordinates [20]. Then, one can perform the calculation in the momentum space by using the Fourier transform, $-\tilde{\Box} + m^2 \to \tilde{k}_\mu\tilde{k}^\mu + m^2$, where $\tilde{k}$ is the momentum measured in the local coordinates. Consequently, it is possible to recover the global coordinates for the covariant result (13).

In the Euclidean geometry, the time component of the four vector k_μ becomes $2\pi n/(-i\beta)$, and the integrals $\int dt$ and $\int dk_0/(2\pi)$ can be replaced by $-i\int d\tau$ and $(-i\beta)^{-1}\sum_n$, respectively [21]. Then, the Euclidean one loop effective action at the finite temperature can be written as