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Relativistic two-body Coulomb–Breit Hamiltonian in an external weak gravitational field

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ABSTRACT

A construction of the Coulomb–Breit Hamiltonian for a pair of fermions, considered as a quantum two-body system, immersed in an arbitrary background gravitational field described by Einstein's General Relativity is presented. Working with Fermi normal coordinates for a freely falling observer in a spacetime region where there are no background sources and ignoring the gravitational back-reaction of the system, the effective Coulomb–Breit Hamiltonian is obtained starting from the S-matrix element corresponding to the one-photon exchange between the charged fermionic currents. The contributions due to retardation are considered up to order $(v/c)^2$ and they are subsequently written as effective operators in the relativistic quantum mechanical Hilbert space of the system. The final Hamiltonian includes effects linear in the curvature and up to order $(v/c)^2$.

1. Introduction

The connection between Einstein's General Relativity (GR), the dynamical theory of spacetime, and Quantum Mechanics (QM) has a long history in physics and more than ever is nowadays a subject of intense research, both from theoretical and experimental perspectives, mainly in connection with possible deviations from GR and/or QM. Until now, experiments have already confirmed that inertia and Newtonian gravity affect quantum particles, mainly electrons and neutrons, in ways that are fully consistent with GR down to distances of the order of 10^{-8} cm. Gravitational-inertial fields leave their mark on particle wave functions in a variety of ways; particularly, they induce quantum phases that have been measured in some of the most renowned experiments on this topic [1–6]. Recently, the purpose of tests of gravity has been mainly focused in determining the validity of the equivalence principle at the level of atoms, molecules, quasi-particles and antimatter. This translates into a need for higher precision tests of GR, and atomic physics together with quantum optics offer some of the most accurate results and promising scenarios.

Most of the experimental tests of GR within the quantum realm arise from predictions on fully covariant wave equations where inertia and gravity appear as external classical fields, providing very valuable information on how Einstein's views carry through into the quantum world. However, many of the initially proposed and tested effects considered mainly one-particle systems. This approach leaves aside the possibility of using the internal quantum structure of atoms or molecules as additional parameters. Nevertheless, this situation is drastically changing recently due to the fast development of matter wave interferometry [7–9].

Broadly speaking, previous work related to the description of atoms in a background gravitational field can be divided in three main branches, though with much interrelation among them: (i) the study of gravitational modifications to atomic spectra of mainly hydrogen-like atoms, which were considered as a quantum reduced mass revolving around some force center [10–21]. A two-body description of the atom can be found in Refs. [22–24]. (ii) Restricted proofs of the equivalence principle for test bodies made of classical electromagnetically interacting particles, within the PPN framework or within the $TH\epsilon\mu$ formalism for static spherically symmetric spacetimes [25–28]. The formulation of a quantum equivalence principle has been studied in Refs. [29,30]. (iii) Tests and studies of the gravitational red shift, where atomic clocks were described as quantum mechanical systems within the $TH\epsilon\mu$ or PPN formalisms [31–37].

In this Letter we consider the effect of a classical background gravitational field, described by GR, upon matter at the atomic level, where the use of QM becomes mandatory. We extend previous work in the following aspects: (i) We will consider the case of hydrogen-like atoms described as two-body systems in the way first discussed in Ref. [23]. The inclusion of the position of the atom as a new degree of freedom may provide additional couplings to probe the effects of gravity upon quantum objects in the region where tidal forces become important. (ii) The background

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gravitational field has no particular spatial symmetry. (iii) Following the steps of Ref. [38], a quantum mechanical relativistic two-body effective potential in the background gravitational field is produced, which takes into account retardation effects arising from the one-photon exchange between the currents. This constitutes the gravitational generalization of the relativistic Coulomb–Breit potential [39].

Most of the general assumptions underlying this work are similar to those of Refs. [10,13,15,23,25,26]: (i) The external gravitational field is well described by GR, so that it satisfies Einstein's field equations. Additionally, we ignore back-reaction effects. (ii) All the interacting particles are influenced by the background gravitational field at the quantum level. (iii) There exists an ideal freely falling observer in a region outside the sources of the background gravitational field. This observer determines that, in his coordinate patch, the metric of the spacetime has the form[1]

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \mathcal{Q}_{\alpha b\beta c} x^b x^c \equiv \eta_{\alpha\beta} + h_{\alpha\beta}(x^0), \tag{1}$$

where

$$\mathcal{Q}_{0a0b} = -\breve{R}_{0a0b}(x^0), \qquad \mathcal{Q}_{0abc} = -\frac{2}{3}\breve{R}_{0abc}(x^0),$$
$$\mathcal{Q}_{abcd} = -\frac{1}{3}\breve{R}_{abcd}(x^0). \tag{2}$$

Here x^a are three spatial coordinates, x^0 is the proper time of the observer and $\breve{R}_{\alpha\beta\gamma\delta}$ is the projection of the background Riemann tensor on the orthonormal tetrad carried by him, which in general depends on the proper time due to the motion of the observer. The weakness of the gravitational interaction induces very small corrections upon the observables, so that in our final results we preserve only quantities up to first order in $\breve{R}_{\alpha\beta\gamma\delta}$. The metric (1) is that of a freely falling observer using Fermi normal coordinates, which are appropriate to describe local experiments performed by inertial observers [40]. (iv) The spatial extension and the time duration of events related to observations in the quantum system are very small compared with the characteristic lengths and times of appreciable changes in the observer's Riemann tensor [15]. This allows well-defined energy levels and the use of time-independent perturbation theory. Within this adiabatic approximation it is possible to ignore the x^0 dependence of the metric and of all the objects constructed from it: the time coordinate becomes just a fixed but arbitrary parameter [26]. (v) Conditions (i) and (iii) imply that the observer determines that his Ricci tensor $\breve{R}^\alpha{}_\beta$ is equal to zero. We also assume that during the measurements performed by the observer there are no particle creation effects due to the gravitational field.

The coupled equations that describe the electromagnetic interaction between spin 1/2 fermions in a background gravitational field, outside the gravitational sources, are

$$i c\hbar\gamma^\mu D_\mu\psi - mc^2\psi = q\gamma^\mu\psi A_\mu,$$
$$D^\mu D_\mu A^\nu = q\bar{\psi}\gamma^\nu\psi \equiv J^\nu, \tag{3}$$

where we have chosen the Lorentz gauge $D_\mu A^\mu = 0$. Here $D_\mu = \partial_\mu - \frac{i}{2}\omega^{BD}{}_\mu J_{BD}$ is the covariant derivative written in general terms, $\omega^{\bar{\alpha}}{}_{\bar{\beta}\mu} = e^{\bar{\alpha}}{}_\delta (D_\mu e_{\bar{\beta}}{}^\delta)$ is the spin connection, and J_{BD} are the Lorentz group generators in the corresponding representation for each field.

[1] Our metric has signature -2, $\eta^{\bar{\alpha}\bar{\beta}} = diag(-1,+1,+1,+1)$, $e_{\bar{\alpha}}{}^\mu$ is the tetrad such that $e_{\bar{\alpha}}{}^\mu \eta^{\bar{\alpha}\bar{\beta}} e_{\bar{\beta}}{}^\nu = g^{\mu\nu}$, $\bar{\alpha} = \{\bar{0},\bar{1},\bar{2},\bar{3}\} = \{\bar{0},\bar{a}\}$ are tetrad indices, while $\mu = \{0,1,2,3\} = \{0,a\}$ are coordinate indices. Also $e_{\bar{\alpha}}{}^\mu e^{\bar{\alpha}}{}_\nu = \delta^\mu_\nu$. The symbol $\stackrel{\circ}{=}$ denotes an equality including at most terms linear in the curvature.

2. The photon propagator

The fundamental quantity required to construct an effective relativistic Hamiltonian describing the electromagnetic interaction between two charges is the curved space Feynman Green function $G^\mu{}_\nu(x,x')$ for the photon [41]. We consider its Hadamard representation because it is valid in all spacetimes [43]. The differential equation for the electromagnetic Feynman Green function is

$$\Box G^\alpha{}_\beta(x,x') = -\kappa^\alpha{}_\beta(x,x')\delta_4(x,x'), \tag{4}$$

where $\Box \equiv g^{\mu\nu}D_\mu D_\nu$, $\kappa^\alpha{}_\beta$ is the parallel propagator bitensor between x' and x, and $\delta_4(x,x')$ is the invariant Dirac delta distribution for our observer. The solution to (4), that vanishes at infinity for arbitrary time, is

$$G_{\alpha\beta}(x,x') = \frac{1}{(2\pi)^2}\left(\frac{\kappa_{\alpha\beta}(x,x')}{\sigma(x,x') + i\epsilon}\right). \tag{5}$$

The expressions for the parallel propagator $\kappa^\alpha{}_\beta(x,x') \equiv e_{\bar{\gamma}}{}^\alpha(x) e^{\bar{\gamma}}{}_\beta(x')$ are[2]

$$\kappa_{00} \stackrel{\circ}{=} \eta_{00} + \frac{1}{2}\left[h_{00}(x) + h_{00}(x')\right],$$
$$\kappa_{0a} \stackrel{\circ}{=} \frac{1}{4}\left[3h_{0a}(x) + h_{0a}(x')\right], \tag{6}$$
$$\kappa_{a0} \stackrel{\circ}{=} \frac{1}{4}\left[h_{0a}(x) + 3h_{0a}(x')\right],$$
$$\kappa_{ab} \stackrel{\circ}{=} \eta_{ab} + \frac{1}{2}\left[h_{ab}(x) + h_{ab}(x')\right]. \tag{7}$$

Here $\sigma(x,x')$ is the Synge world function, given by

$$\sigma(x,x') \stackrel{\circ}{=} \frac{1}{2}\eta_{\mu\nu}(x-x')^\mu(x-x')^\nu + \frac{1}{2}\mathcal{Q}_{\mu a\nu b}\mathcal{A}^{ab}(x-x')^\mu(x-x')^\nu \tag{8}$$

with

$$\mathcal{A}^{ab} \equiv \left[\frac{1}{2}\left(x'^a x^b + x^a x'^b\right) + \frac{1}{3}(x-x')^a(x-x')^b\right]. \tag{9}$$

Note that expression (5) contains terms of higher order than first in the curvature, arising from the denominator $\sigma + i\epsilon$. We will keep the complete singular structure of the Green function because, in the following calculations, we will require expressions for the corresponding poles up to first order in $h_{\mu\nu}$. Let us remark that $\sigma(x,x') = \sigma(\mathbf{x},\mathbf{x}',x^0 - x'^0)$.

3. The one-photon interaction

In order to incorporate gravitational corrections into the electromagnetic interaction between two charged particles, we will use the S-matrix method [41] as described in Ref. [38], but generalized to a slightly curved space [42]. We need to evaluate the S-matrix element corresponding to the exchange of one photon between the two fermionic currents located at x_2 and x_1, shown in Fig. 1, which is given by

$$S^{(1)}_{fi} = \frac{1}{c\hbar}\int J^{\mu(2)}_{fi}(x_2) G_{\mu\nu}(x_2,x_1) J^{\nu(1)}_{fi}(x_1)\, d^4V_2\, d^4V_1, \tag{10}$$

where $G_{\mu\nu}(x_2,x_1)$ is the electromagnetic Feynman Green function (5) and $d^4V \equiv \sqrt{-g(x)}\,d^4x$ is the invariant volume element

[2] The orthonormal tetrad corresponding to the metric (1) is: $e_{\bar{0}0} = -1 - \frac{1}{2}h_{00}$, $e_{\bar{0}a} = -\frac{1}{4}h_{0a}$, $e_{\bar{a}0} = \frac{3}{4}h_{0a}$, $e_{\bar{a}b} = \eta_{ab} + \frac{1}{2}h_{ab}$ [15].