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Kohn's theorem and Galilean symmetry

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ABSTRACT

The relation between the separability of a system of charged particles in a uniform magnetic field and Galilean symmetry is revisited using Duval's "Bargmann framework". If the charge-to-mass ratios of the particles are identical, $e_a/m_a = \epsilon$ for all particles, then the Bargmann space of the magnetic system is isometric to that of an anisotropic harmonic oscillator. Assuming that the particles interact through a potential which only depends on their relative distances, the system splits into one representing the center of mass plus a decoupled internal part, and can be mapped further into an isolated system using Niederer's transformation. Conversely, the manifest Galilean boost symmetry of the isolated system can be "imported" to the oscillator and to the magnetic systems, respectively, to yield the symmetry used by Gibbons and Pope to prove the separability. For vanishing interaction potential the isolated system is free and our procedure endows all our systems with a hidden Schrödinger symmetry, augmented with independent internal rotations. All these properties follow from the cohomological structure of the Galilei group, as explained by Souriau's *"décomposition barycentrique"*.

1. Introduction

Kohn's theorem [1], commonly but vaguely ascribed to Galilean invariance, says that a system of charged particles in a uniform magnetic field can be decomposed into center-of-mass and relative motion if the charge/mass ratios are identical,

$$e_a/m_a = \epsilon = \text{const.} \tag{1}$$

The term "Galilean invariance" has been recently been criticized by Gibbons and Pope [2], though, who argue that their symmetry transformation $\vec{x} \to \vec{x} + \vec{a}(t)$ is not of the usual Galilean form $\vec{x} \to \vec{x} + \vec{b}t$, and belongs rather to the Newton–Hooke group.

In this Note we show that the two, apparently contradictory, statements *can* be conciliated: $\vec{a}(t)$ *is* a Galilean boost, — but it acts in a way which is different form the usual one. Separability *does follow therefore from "abstract" Galilean invariance — as it does from Newton–Hooke symmetry also*. In detail, we show that when (1) holds the Bargmann space of the magnetic-background system is conformally related to an isolated system with ordinary boost symmetry, and "importing" it guarantees the existence of a rest frame also for the magnetic-background. The "imported boost" coincides with the symmetry used by Gibbons and Pope [2].

In the absence of an interaction potential, the system carries, moreover, a "hidden" Schrödinger symmetry obtained by "importing" that of a free system, augmented with internal rotations. Our results shed new light on Kohn's theorem and generalize Souriau's "*décomposition barycentrique*" [3].

2. A "relativistic" proof of Kohn's theorem

We demonstrate our statements in the Kaluza–Klein-type framework [4] which says that the *null geodesics of a manifold in* $d+2$ *dimensions* with Lorentz metric,

$$ds^2 = d\vec{x}^2 + 2\,dt\,ds - \frac{2U}{m}(\vec{x}, t)\,dt^2 \tag{2}$$

project, for a particle in $(d+1)$-dimensional non-relativistic space–time with coordinates $(\vec{x}, t)$, according to Newton's equations, $m\ddot{\vec{x}} = -\vec{\nabla}U$. The generalization of (2) to N particles in d dimensions in a potential U is provided by the $(Nd+2)$ dimensional metric [4],

$$\sum_{a=1}^{N} \frac{m_a}{m}\,d\vec{x}_a^2 + 2\,dt\,ds - \frac{2U}{m}\,dt^2 \quad \text{where } m = \sum_{a=1}^{N} m_a. \tag{3}$$

A remarkable property of the metric (2) is that it defines a preferential Newton–Cartan structure [5] on non-relativistic space-time obtained by projecting out the "vertical" direction generated by the lightlike vector ∂_s [4]. In the quadratic case $U = \pm\frac{1}{2}\omega^2\vec{x}^2$, (2) describes, from the mechanical point of view, an [attractive of repulsive] harmonic oscillator [4]. In a relativistic language (2) is

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a pp-wave, and the quotient is Newton–Hooke space–time [2,5], which carries a Newton–Hooke symmetry, represented by the isometries of the metric [4,2].

But the metric (3) is just one example of a "Bargmann" space-time, whose characteristic feature is that it carries a covariantly constant lightlike vector [4]. More generally, the metric can also accommodate a vector potential [6]: the projections of the null geodesics of

$$ds^2 = d\vec{x}^2 + 2\,dt\left(ds + \frac{e}{m}\vec{A}(\vec{x})\cdot d\vec{x}\right) - \frac{2e}{m}U(\vec{x})\,dt^2 \tag{4}$$

satisfy the usual [Lorentz] equations of motion of a non-relativistic particle in a (static) "electromagnetic" field $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}U$.

A remarkable feature is that, in the plane, the isotropic oscillator metric (2) with $U = \frac{1}{2}\omega^2\vec{x}^2$ is indeed *equivalent* to the "magnetic" metric (4) with vector potential $A_i = -\frac{B}{2}\epsilon_{ij}x^j$, used to describe the motion in a uniform magnetic field B perpendicular to the plane.[2] Switching to a rotating frame,

$$\vec{X} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = R_B\vec{x} \equiv \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}; \qquad \Omega = \frac{eB}{2m}, \tag{5}$$

completed with $T = t$ and $S = s$, carries the magnetic metric into that of the oscillator. (This is just the familiar Larmor trick in a new guise.) N particles in the plane with electric charges e_a are described by adding to the metric (3) $2\,dt\sum_a(e_a/m)\vec{A}_a\cdot d\vec{x}_a$, where $A_a^i = -(B/2)\epsilon^i{}_j x_a^j$.

The generalization to N particles being straightforward, we restrict ourselves henceforth to two charged planar particles in a constant magnetic field. With the same choice of gauge for $\vec{A}$ as above, we hence consider the $(2\times 2+1+1 = 6)$-dimensional metric

$$\sum_a \frac{m_a}{m}d\vec{x}_a^2 + 2\,dt\,ds - B\sum_a\frac{e_a}{m}\left(x_a^2\,dx_a^1 - x_a^1\,dx_a^2\right)dt - \frac{2V}{m}dt^2, \tag{6}$$

where we have included an interaction potential $V \equiv V(|\vec{x}_a - \vec{x}_b|)$ and dropped the external trapping potential U for simplicity. Then, applying (5) to each vector $\vec{x}_a$ $(a = 1, 2)$ yields

$$\sum_a\frac{m_a}{m}d\vec{X}_a^2 + 2\,dT\,dS - \frac{2V}{m}dT^2 + \frac{\Omega}{m}\sum_a\left[(m_a\Omega - e_aB)\vec{X}_a^2\right]dT^2 + \frac{1}{m}\sum_a\left[(2m_a\Omega - e_aB)\left(X_a^1\,dX_a^2 - X_a^2\,dX_a^1\right)\right]dT.$$

Our clue is now that *if the particles have the same charge to mass ratios*, (1), then, choosing the rotation frequency as $\Omega = \epsilon B/2$ carries the constant-magnetic-field-metric, (6), into

$$\sum_a\frac{m_a}{m}d\vec{X}_a^2 + 2\,dT\,dS - \frac{2}{m}\left(\frac{\omega^2}{2}\sum_a m_a\vec{X}_a^2 + V\right)dT^2, \qquad \omega^2 = \epsilon^2\frac{B^2}{4}, \tag{7}$$

which is the metric for an *anisotropic* oscillator in $d = 2+2$ dimensions, augmented with the potential V.[3] The two-particle metric (7) plainly decomposes into center-of-mass and relative coordinates. Putting

$$\vec{X}_0 = \frac{m_1\vec{X}_1 + m_2\vec{X}_2}{m}, \qquad \vec{Y} = \frac{\sqrt{m_1m_2}}{m^2}(\vec{X}_1 - \vec{X}_2) \tag{8}$$

and calling $V(|\vec{Y}|m^2(m_1m_2)^{-1/2})$ again $V(|\vec{Y}|)$ with some abuse of notations (7) is indeed written as

$$\left\{d\vec{X}_0^2 - \omega^2\vec{X}_0^2\,dT^2\right\} + \left\{d\vec{Y}^2 - \left(\omega^2\vec{Y}^2 + \frac{2V(|\vec{Y}|)}{m}\right)dT^2\right\} + 2\,dT\,dS. \tag{9}$$

The first curly bracket here clearly describes the center-of-mass which behaves as a planar particle of mass m in an attractive oscillator field, to which the "internal vector" $\vec{Y}$ adds two more dimensions, interpreted as an "internal oscillator" with an interaction potential. Note that the "external" and "internal" oscillators have identical frequencies ω and also that the anisotropic oscillator became isotropic when expressed in the new coordinates. The null geodesics of the metric (9) project to the decoupled system of planar oscillators

$$\frac{d^2\vec{X}_0}{dT^2} + \omega^2\vec{X}_0 = 0, \qquad \frac{d^2\vec{Y}}{dT^2} + \omega^2\vec{Y} + \frac{1}{m}\vec{\nabla}_Y V = 0. \tag{10}$$

The center-of-mass, $\vec{X}$, performs an elliptic "deferent" motion around the origin, to which $\vec{Y}$ adds an "epicycle" with the same oscillator frequency, plus some internal interaction. Transforming back to the magnetic background, we have

$$\left\{d\vec{x}_0^2 + \epsilon B(\vec{x}_0\times d\vec{x}_0)\,dt\right\} + \left\{d\vec{y}^2 + \epsilon B(\vec{y}\times d\vec{y})\,dt - 2\frac{V(|\vec{y}|)}{m}dt^2\right\} + 2\,dt\,ds, \tag{11}$$

$$\ddot{x}_0^i = -2\omega\epsilon^{ij}\dot{x}_0^j, \qquad \ddot{y}^i = -2\omega\epsilon_i{}^j\dot{y}^j - \partial_{y^i}V, \tag{12}$$

where $\vec{x}_0 = R_B^{-1}\vec{X}$ is the magnetic center-of-mass and $\vec{y} = R_B^{-1}\vec{Y}$ is the internal coordinate.

The decomposition (9) [or (10)] allows us to infer that the system admits *two independent and separately conserved angular momenta*, since one can consider *independent external and internal rotations*,

$$\begin{aligned} &\vec{X}_0 \to R_{ext}\vec{X}_0, && \vec{Y}\to\vec{Y}, && L_0 = m\vec{X}\times\dot{\vec{X}}, \\ &\vec{X}_0\to\vec{X}_0, && \vec{Y}\to R_{int}\vec{Y}, && L_{int} = m\vec{Y}\times\dot{\vec{Y}}, \end{aligned} \tag{13}$$

where R_{int} and R_{ext} are planar rotation matrices. The first rotation corresponds to rotating the center of mass alone, and the second corresponds to rotating it around its center of mass. The (separate) conservations of the two angular momenta can be checked directly using the equations of motion (10) or (12).

3. Mapping to an isolated system and hidden Schrödinger symmetry

Another remarkable feature of the metric (2) [with $\vec{x}\rightsquigarrow\vec{X}$, $t\rightsquigarrow T$] is that, in the quadratic case $2U = \pm\omega^2\vec{X}^2$ and for uniform $B = B(t)$, it is *conformally flat* [4,6]. For $B = $ const, lifting Niederer's transformation [9] to Bargmann space according to

$$\vec{\Xi} = \frac{\vec{X}}{\cos\omega T}, \qquad \tau = \frac{\tan\omega T}{\omega}, \qquad \Sigma = s - \frac{\omega}{2}\vec{X}^2\tan\omega T \tag{14}$$

maps in fact each half-period of the oscillator conformally into the free metric [9], $d\vec{\Xi}^2 + 2\,d\tau\,d\Sigma = \cos^{-2}\omega T(d\vec{X}^2 + 2\,dT\,dS - 2U\,dT^2)$. Generalizing (14), we observe that

$$\vec{\Xi}_a = \frac{\vec{X}_a}{\cos\omega T}, \qquad \tau = \frac{\tan\omega T}{\omega}, \qquad \Sigma = s - \frac{\omega}{2}\left(\sum_a\frac{m_a}{m}\vec{X}_a^2\right)\tan\omega T \tag{15}$$

[2] For simplicity, we took $B(t) = B = $ const and work in the plane.

[3] The relation of the (non-commutative) Landau problem with an anisotropic harmonic oscillator has also been studied [7].