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Comparing scalar–tensor gravity and $f(R)$-gravity in the Newtonian limit

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ABSTRACT

Recently, a strong debate has been pursued about the Newtonian limit (i.e. small velocity and weak field) of fourth order gravity models. According to some authors, the Newtonian limit of $f(R)$-gravity is equivalent to the one of Brans–Dicke gravity with $\omega_{BD}=0$, so that the PPN parameters of these models turn out to be ill-defined. In this Letter, we carefully discuss this point considering that fourth order gravity models are dynamically equivalent to the O'Hanlon Lagrangian. This is a special case of scalar–tensor gravity characterized only by self-interaction potential and that, in the Newtonian limit, this implies a non-standard behavior that cannot be compared with the usual PPN limit of General Relativity. The result turns out to be completely different from the one of Brans–Dicke theory and in particular suggests that it is misleading to consider the PPN parameters of this theory with $\omega_{BD}=0$ in order to characterize the homologous quantities of $f(R)$-gravity. Finally the solutions at Newtonian level, obtained in the Jordan frame for an $f(R)$-gravity, reinterpreted as a scalar–tensor theory, are linked to those in the Einstein frame.

1. Introduction

Recently, several authors claimed that higher order theories of gravity, in particular $f(R)$-gravity [1], are characterized by an ill-defined behavior in the Newtonian regime. In a series of papers [2], it is discussed that higher order theories violate experimental constraints of General Relativity (GR) since a direct analogy between $f(R)$-gravity and Brans–Dicke gravity [3] gives the Brans–Dicke characteristic parameter, in metric formalism, $\omega_{BD}=0$ while it should be $\omega_{BD}\to\infty$ to recover the standard GR. Actually despite the calculation of the Newtonian limit of $f(R)$, directly performed in the Jordan frame, have showed that this is not the case [4,5], it remains to clarify why the analogy with Brans–Dicke gravity seems to fail its predictions also if one is assuming $f(R)\simeq R^{1+\epsilon}$ with $\epsilon\to 0$. The shortcoming could be overcome once the correct analogy between $f(R)$-gravity and the scalar–tensor framework is taken into account.

The action of the Brans–Dicke gravity, in the Jordan frame, reads:

$$\mathcal{A}^{BD}_{JF}=\int d^4x\sqrt{-g}\left[\phi R+\omega_{BD}\frac{\phi_{;\alpha}\phi^{;\alpha}}{\phi}+\mathcal{X}\mathcal{L}_m\right], \tag{1}$$

where there is a generalized kinetic term and no potential is present. On the other hand, considering a generic function $f(R)$ of the Ricci scalar R, one has:

$$\mathcal{A}^{f(R)}_{JF}=\int d^4x\sqrt{-g}\left[f(R)+\mathcal{X}\mathcal{L}_m\right]. \tag{2}$$

In both cases, $\mathcal{X}=\frac{8\pi G}{c^4}$ is the standard Newton coupling, $\mathcal{L}_m$ is the perfect fluid matter Lagrangian and g is the determinant of the metric.

As is said above, $f(R)$-gravity can be re-interpreted as a scalar–tensor theory by introducing a suitable scalar field ϕ which non-minimally couples with the gravity sector. It is important to remark that such an analogy holds in a formalism in which the scalar field displays no kinetic term but is characterized by means of a self-interaction potential which determines the dynamics (*O'Hanlon Lagrangian*) [6]. This consideration, therefore, implies that the scalar field Lagrangian, equivalent to the purely geometrical $f(R)$ one, turns out to be different with respect to the above ordinary Brans–Dicke definition (1). This point represents a crucial aspect of our analysis. In fact, as we will see below, such a difference will imply completely different results in the Newtonian limit of the two models and, consequently, the impossibility to compare

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predictions coming from the PPN approximation of Brans–Dicke models to those coming from $f(R)$-gravity.

The layout of the Letter is the following. In Section 2, we discuss the solutions in the Newtonian limit of $f(R)$-gravity by using the analogies with the O'Hanlon theory. Section 3 is devoted to the analysis of the solutions in the limit $f(R) \to R$ and the interpretation of PPN parameters γ, β. Conformal transformations and the solutions in the Newtonian limit approximation are considered in Section 4. Concluding remarks are drawn in Section 5.

2. The Newtonian limit of $f(R)$-gravity by O'Hanlon theory

Before starting with our analysis, let us remind that the field equations in metric formalism, coming from $f(R)$-gravity, are

$$H_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}T_{\mu\nu} \tag{3}$$

which have to be solved together to the trace equation

$$\Box f'(R) + \frac{f'(R)R - 2f(R)}{3} = \frac{\mathcal{X}}{3}T. \tag{4}$$

Let us notice that this last expression assigns the evolution of the Ricci scalar as a dynamical quantity. Here, $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy–momentum tensor of matter, while $T = T^{\sigma}{}_{\sigma}$ is the trace, $f'(R) = \frac{df(R)}{dR}$. The convention for Ricci's tensor is $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$ while for the Riemann tensor is $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \cdots$.

The affine connections are the Christoffel symbols of the metric: $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$. The adopted signature is $(+---)$.

On the other hand, the so-called O'Hanlon Lagrangian [6] can be written as

$$\mathcal{A}^{OH}_{JF} = \int d^4x\sqrt{-g}\left[\phi R - V(\phi) + \mathcal{X}\mathcal{L}_m\right], \tag{5}$$

where $V(\phi)$ is the self-interaction potential. Field equations are obtained by varying Eq. (5) with respect to both $g_{\mu\nu}$ and ϕ which now represent the dynamical variables. Thus, one obtains

$$\phi G_{\mu\nu} + \frac{1}{2}V(\phi)g_{\mu\nu} - \phi_{;\mu\nu} + g_{\mu\nu}\Box\phi = \mathcal{X}T_{\mu\nu}, \tag{6}$$

$$R - \frac{dV(\phi)}{d\phi} = 0, \tag{7}$$

$$\Box\phi - \frac{1}{3}\left[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right] = \frac{\mathcal{X}}{3}T, \tag{8}$$

where we have displayed the field equation for ϕ. Eq. (8) is a combination of the trace of (6) and (7). $f(R)$-gravity and O'Hanlon gravity can be mapped one into the other considering the following equivalences

$$\phi = f'(R), \tag{9}$$

$$V(\phi) = f'(R)R - f(R), \tag{10}$$

$$\phi\frac{dV(\phi)}{d\phi} - 2V(\phi) = 2f(R) - f'(R)R \tag{11}$$

and supposing that the Jacobian of the transformation $\phi = f'(R)$ is non-vanishing. Henceforth we can consider, instead of Eqs. (3)–(4), a new set of field equations determined by the equivalence between the O'Hanlon gravity and the $f(R)$-gravity:

$$\phi R_{\mu\nu} - \frac{1}{6}\left[V(\phi) + \phi\frac{dV(\phi)}{d\phi}\right]g_{\mu\nu} - \phi_{;\mu\nu} = \mathcal{X}\Sigma_{\mu\nu}, \tag{12}$$

$$\Box\phi - \frac{1}{3}\left[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right] = \frac{\mathcal{X}}{3}T, \tag{13}$$

where $\Sigma_{\mu\nu} \doteq T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}$.

Let us, now, calculate the Newtonian limit of Eqs. (12)–(13). To perform this calculation, the metric tensor $g_{\mu\nu}$ and the scalar field ϕ have to be perturbed with respect to the background. After, one has to search for solutions at the $(v/c)^2$ order in term of the metric and the scalar field entries. It is

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + g^{(2)}_{00} & \vec{0}^T \\ \vec{0} & -\delta_{ij} + g^{(2)}_{ij} \end{pmatrix}, \tag{14}$$

$$\phi \sim \phi^{(0)} + \phi^{(2)}. \tag{15}$$

The differential operators turn out to be approximated as

$$\Box \approx \partial_0^2 - \Delta \quad \text{and} \quad {}_{;\mu\nu} \approx \partial^2_{\mu\nu}, \tag{16}$$

since time derivatives increase the degree of perturbation, they can be discarded [4]. From a physical point of view, this position holds since Newtonian limit implies also the slow motion.

Actually in order to simplify calculations, we can exploit the gauge freedom that is intrinsic in the metric definition. In particular, we can choose the harmonic gauge $g^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma} = 0$ so that the components of Ricci tensor reduces to

$$\begin{cases} R^{(2)}_{00} = \frac{1}{2}\triangle g^{(2)}_{00}, \\ R^{(3)}_{0i} = 0, \\ R^{(2)}_{ij} = \frac{1}{2}\triangle g^{(2)}_{ij}. \end{cases} \tag{17}$$

Accordingly, we develop the self-interaction potential at second order. In particular, the quantities in Eqs. (12) and (13) read:

$$V(\phi) + \phi\frac{dV(\phi)}{d\phi} \simeq V\left(\phi^{(0)}\right) + \phi^{(0)}\frac{dV(\phi^{(0)})}{d\phi} + \left[\phi^{(0)}\frac{d^2V(\phi^{(0)})}{d\phi^2} + 2\frac{dV(\phi^{(0)})}{d\phi}\right]\phi^{(2)}, \tag{18}$$

$$\phi\frac{dV(\phi)}{d\phi} - 2V(\phi) \simeq \phi^{(0)}\frac{dV(\phi^{(0)})}{d\phi} - 2V\left(\phi^{(0)}\right) + \left[\phi^{(0)}\frac{d^2V(\phi^{(0)})}{d\phi^2} - \frac{dV(\phi^{(0)})}{d\phi}\right]\phi^{(2)}. \tag{19}$$

Field equations (12)–(13), solved at 0th order of approximation, provide the two solutions

$$V\left(\phi^{(0)}\right) = 0 \quad \text{and} \quad \frac{dV(\phi^{(0)})}{d\phi} = 0 \tag{20}$$

which fix the 0th order terms of the self-interaction potential; therefore we have

$$-\phi\frac{dV(\phi)}{d\phi} \simeq -\phi^{(0)}\frac{d^2V(\phi^{(0)})}{d\phi^2}\phi^{(2)} \doteq 3m^2\phi^{(2)}, \tag{21}$$

where the constant factor m^2 can be easily interpreted as a mass term as will become clearer in the following analysis (see also [9]). Now, taking into account the above simplifications, we can rewrite the field equations at the $(v/c)^2$ order in the form:

$$\triangle g^{(2)}_{00} = \frac{2\mathcal{X}}{\phi^{(0)}}\Sigma^{(0)}_{00} - m^2\frac{\phi^{(2)}}{\phi^{(0)}}, \tag{22}$$

$$\triangle g^{(2)}_{ij} = \frac{2\mathcal{X}}{\phi^{(0)}}\Sigma^{(0)}_{ij} + m^2\frac{\phi^{(2)}}{\phi^{(0)}}\delta_{ij} + 2\frac{\phi^{(2)}_{,ij}}{\phi^{(0)}}, \tag{23}$$

$$\triangle\phi^{(2)} - m^2\phi^{(2)} = -\frac{\mathcal{X}}{3}T^{(0)}. \tag{24}$$