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Spherical collapse model with and without curvature

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ABSTRACT

We investigate a spherical collapse model with and without the spatial curvature. We obtain the exact solutions of dynamical quantities such as the ratio of the scale factor to its value at the turnaround epoch and the ratio of the overdensity radius to its value at the turnaround time with general cosmological parameters. The exact solutions of the overdensity at the turnaround epoch for the different models are also obtained. Thus, we are able to obtain the nonlinear overdensity at any epoch for the given model. We obtain that the nonlinear overdensity of the Einstein de Sitter (EdS) universe at the virial epoch is $18\pi^2(\frac{1}{2\pi}+\frac{3}{4})^2 \simeq 147$ instead of the well-known value $18\pi^2 \simeq 178$. In the open universe, perturbations are virialized earlier than in the flat one and thus clusters are denser at the virial epoch. Also the critical density threshold of EdS universe from the linear theory at the virialized epoch is obtained as $\frac{3}{20}(9\pi+6)^{\frac{2}{3}} \simeq 1.58$ instead of $\frac{3}{20}(12\pi)^{\frac{2}{3}} \simeq 1.69$. This value is same for the close and the open universes. We find that the observed quantities at high redshifts are less sensitive between different models. Even though the low redshift cluster shows the stronger model dependence than high redshift one, the differences between models might be still too small to be distinguished by observations if the curvature is small. From these analytic forms of dynamical quantities, we are able to estimate the abundances of both virialized and non-virialized clusters and the temperature and luminosity functions at any epoch. The current concordance model prefers the almost flat universe and thus the above results might be restricted by the academic interests only. However, the mathematical structure of the evolution equations of physical quantities for the curved space is identical with that for the flat universe including the dark energy with the equation of state $\omega_{\rm de} = -\frac{1}{3}$. Thus, we might be able to extend these analytic solutions to the general dark energy model and they will provide the useful tools for probing the properties of dark energy.

1. Spherical collapse model

Background evolution equations of the physical quantities in a FRW universe with the matter are given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm m} - \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\rm cr}, \tag{1.1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{\rm m}, \tag{1.2}$$

$$\dot{\rho}_{\rm m} + 3\left(\frac{\dot{a}}{a}\right)\rho_{\rm m} = 0, \tag{1.3}$$

where a is the scale factor, $\rho_{\rm m}$ is the energy density of the matter, $\rho_{\rm cr}$ is the critical energy density, and k is chosen to be $+1$, 0, or -1 for spaces of constant positive, zero, or negative spatial curvatures, respectively. In terms of the ratio of the matter density to the critical density $\Omega_{\rm m}$, the above Friedmann equation (1.1) becomes

$$\frac{k}{H^2a^2} \equiv \Omega_{\rm k} = \Omega_{\rm m} - 1, \tag{1.4}$$

which is valid for all times.

We consider a spherical perturbation in the matter density. $\rho_{\rm cluster}$ is the matter density within the spherical overdensity radius R. The flatness condition is not held because of the perturbation in the matter. Thus, we have another set of equations governing the dynamics of the spherical perturbation [1]:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho_{\rm cluster}, \tag{1.5}$$

$$\dot{\rho}_{\rm cluster} + 3\left(\frac{\dot{R}}{R}\right)\rho_{\rm cluster} = 0, \tag{1.6}$$

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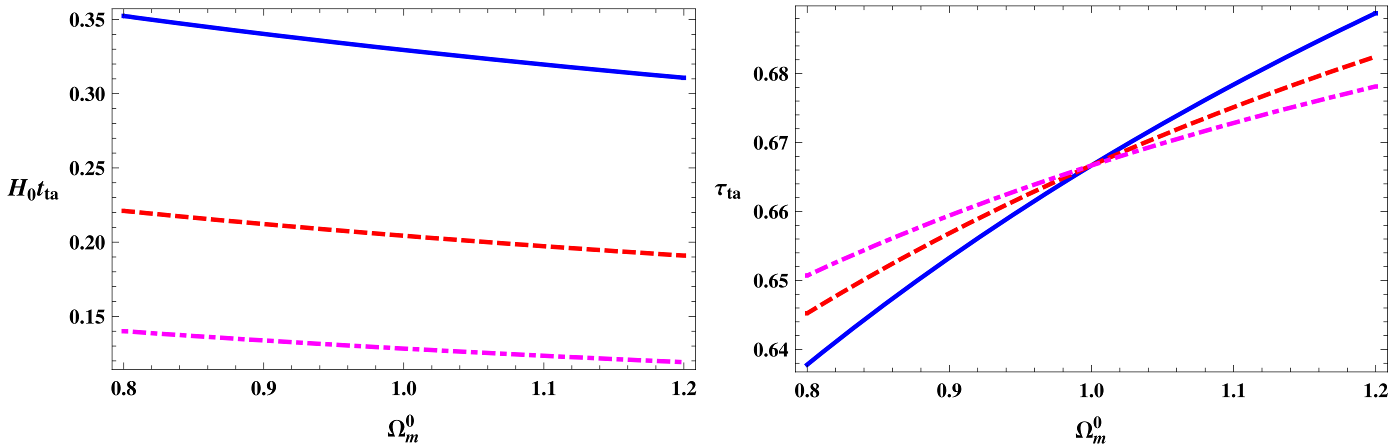

Fig. 1. t_{ta} and τ_{ta} for the different values of z_{ta}. (Left) $H_0 t_{ta}$ versus Ω_m^0 for the different values of $z_{ta} = 0.6$, 1.2, and 2.0 (from top to bottom). (Right) τ_{ta} versus Ω_m^0 for the same values of z_{ta} as in the left panel.

where $\rho_{cluster}$ is the energy density of the clustering matter. The radius of the overdensity R evolves slower than the scale factor a and reaches its maximum size R_{ta} at the turnaround epoch z_{ta} and then the system begins to collapse.

Cosmological parameters and the curvature of the Universe can be constrained from the growth of large scale structure and the abundance of rich clusters of galaxies. There have been numerous works related to this [2–12]. Most of them reach to the similar conclusions based on the conventional approximate solutions of the background scale factor and of the overdensity radius. It is natural to expect that the correct values for the virial radius and the nonlinear overdensity obtained from the exact solutions might be different from those obtained from the conventional approximate solutions. We investigate this.

Now we adopt the notations in Ref. [11] to investigate the evolutions of a and R

$$x = \frac{a}{a_{ta}}, \tag{1.7}$$

$$y = \frac{R}{R_{ta}}, \tag{1.8}$$

where a_{ta} and R_{ta} are the scale factor and the radius at z_{ta}, respectively. Then Eqs. (1.1) and (1.5) are rewritten as

$$\frac{dx}{d\tau} = \sqrt{x^{-1} - Q_{ta}^{-1}}, \tag{1.9}$$

$$\frac{d^2 y}{d\tau^2} = -\frac{1}{2}\zeta y^{-2}, \tag{1.10}$$

where $d\tau = H(x_{ta})\sqrt{\Omega_m(x_{ta})}\,dt \equiv H_{ta}\sqrt{\Omega_{mta}}\,dt$, $Q_{ta} = \frac{\Omega_m}{\Omega_k}|_{z_{ta}} \equiv \frac{\Omega_{mta}}{\Omega_{kta}} = \frac{\Omega_{mta}}{\Omega_{mta}-1} = \frac{\Omega_m^0}{\Omega_m^0-1}(1+z_{ta})$, $\zeta = \frac{\rho_{cluster}}{\rho_m}|_{z_{ta}}$, and $x_{ta} \equiv x(z_{ta}) = 1$ from Eq. (1.7). Ω_m^0 and Ω_k^0 represent the present values of the energy density contrasts of the matter and the curvature term, respectively. Eqs. (1.9) and (1.10) can be solved analytically.

The analytic solution of Eq. (1.9) is given by

$$\int_0^x \frac{dx'}{\sqrt{x'^{-1} - Q_{ta}^{-1}}} = \int_0^\tau d\tau'$$

$$\Rightarrow \quad \frac{2}{3}x^{\frac{3}{2}} F\left[\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{x}{Q_{ta}}\right] = \tau, \tag{1.11}$$

where F is the hypergeometric function and we use the boundary condition $x = 0$ when $\tau = 0$ (see Appendix A for details). From this equation, the exact turnaround time τ_{ta} is given by

$$\tau_{ta} = \frac{2}{3}F\left[\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{\Omega_m^0 - 1}{\Omega_m^0}(1+z_{ta})^{-1}\right]$$
$$= H_{ta}\sqrt{\Omega_{mta}}\,t_{ta} = H_0\sqrt{\Omega_m^0}\,(1+z_{ta})^{\frac{3}{2}} t_{ta}, \tag{1.12}$$

where we use the fact that $x_{ta} = 1$, the relations $Q_{ta} = \frac{\Omega_m^0}{\Omega_m^0-1} \times (1+z_{ta})$, and $\tau = H_{ta}\sqrt{\Omega_{mta}}\,t$. This exact analytic form of the turnaround time will be used to investigate the other quantities.

As expected, τ_{ta} (t_{ta}) depends on Ω_m^0 (*i.e.* Ω_k^0) and z_{ta} as given in Eq. (1.12). We show these properties of τ_{ta} (t_{ta}) in Fig. 1. In the left panel of Fig. 1, we show the dependence of t_{ta} (normalized by multiplying with H_0) on Ω_m^0 for the different values of z_{ta} models. The solid, dashed, and dot-dashed lines (from top to bottom) correspond to $z_{ta} = 0.6$, 1.2, and 2.0, respectively. Eq. (1.11) is the evolution of the background scale factor a and we can interpret it as the age of the Universe is a decreasing function of Ω_m^0. Larger Ω_m^0 implies faster deceleration, which corresponds to a more rapidly expanding universe early on. Also larger z_{ta} means the earlier formation of the structure and thus gives the smaller t_{ta}. We also show the Ω_m^0 dependence of τ_{ta} for the values of z_{ta} in the right panel of Fig. 1. Because $\tau_{ta} = H_{ta}\sqrt{\Omega_{mta}}t_{ta}$, τ_{ta} becomes larger for the larger values of Ω_m^0.

The exact analytic solution of y also can be obtained as (see Appendix A)

$$\text{ArcSin}[\sqrt{y}\,] - \sqrt{y(1-y)} = \sqrt{\zeta}\,\tau, \quad \text{when } \tau \leqslant \tau_{ta}, \tag{1.13}$$

$$\sqrt{y(1-y)} - \text{ArcSin}[\sqrt{y}\,] + \frac{\pi}{2} = \sqrt{\zeta}\,(\tau - \tau_{ta}),$$
$$\text{when } \tau \geqslant \tau_{ta}, \tag{1.14}$$

where τ and τ_{ta} are given in Eqs. (1.11) and (1.12). ζ can be obtained from this analytic solution (1.13) (or equally from Eq. (1.14)) by using the fact that $y_{ta} = 1$

$$\zeta = \left(\frac{\pi}{2\tau_{ta}}\right)^2$$
$$= \left(\frac{3\pi}{4}\right)^2 \left(F\left[\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{(\Omega_m^0 - 1)}{\Omega_m^0}(1+z_{ta})^{-1}\right]\right)^{-2}, \tag{1.15}$$

where we use Eq. (1.12). When $\Omega_m^0 = 1$, $F[\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 0] = 1$ and thus $\zeta = (\frac{3\pi}{4})^2$. This factor $(\frac{3\pi}{4})^2$ is the well-known value of ζ for the Einstein de Sitter (EdS) universe ($\Omega_m = 1$) [1,13]. The general value of ζ for open or closed Universe is given by Eq. (1.15). We show the behavior of ζ in Fig. 2.