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Quantum scale invariance on the lattice

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ABSTRACT

We propose a scheme leading to a non-perturbative definition of lattice field theories which are scale-invariant on the quantum level. A key idea of the construction is the replacement of the lattice spacing by a propagating dynamical field — the dilaton. We describe how to select non-perturbatively the phenomenologically viable theories where the scale invariance is broken spontaneously. Relation to gravity is also discussed.

1. Introduction

It is believed that the scale invariance, existing in *classical* field theories without dimensionfull parameters, is generically broken in quantum field theory (for a review see [1]). The known exceptions include finite theories such as $N = 4$ super-Yang–Mills [2] or other classes of supersymmetric theories [3], which, however, are lacking an immediate phenomenological relevance. The standard regularisations of the ultraviolet infinities introduce one type of mass scale or another and this scale makes its way into the renormalized theory. The explicit breaking of scale invariance (SI) or its anomalous breaking by quantum effects leads to severe fine-tuning problems, facing the realistic models of particle physics. One of them is the problem of stability of the weak scale against quadratic quantum corrections and the second one is the cosmological constant problem, associated with quartic divergences.

A quantum field theory (gravity included) with *exact*, but spontaneously broken SI would solve the above-mentioned problems. The scale-invariance, existing on quantum level, forbids quadratic or quartic divergences, exactly in a way the gauge invariance keeps the photon massless in the Quantum Electrodynamics. The spontaneous breaking of the dilatational symmetry would introduce all different mass scales, observed in Nature (including the mass of the Higgs boson, the QCD scale Λ and alike, the Newtons gravity constant, etc.) through the vacuum expectation value (vev) of the dilaton field. Moreover, a combination of the ideas of SI with those of unimodular gravity [4–6] leads to a possible explanation of primordial inflation and of late acceleration of the universe [7].

In [8] a perturbative way for construction of a new class of *effective* quantum theories, where SI is exact, but broken spontaneously, was suggested. The similar procedure for keeping the local conformal symmetry intact at the quantum level was proposed earlier in [9]. The basic idea is as follows. To have a quantum theory which is scale invariant the renormalisation procedure must not introduce any dimensionfull parameter. This may be achieved, in dimensional regularisation of [10], by replacing the 't Hooft–Veltman normalization scale μ by an appropriate combination of dynamical quantum fields in such a way that the SI is preserved in any space–time dimension $d = 4 - 2\epsilon$. Clearly, this introduces new interactions when $\epsilon \neq 0$. Though their strength is suppressed by ϵ, they leave a trace in the renormalized theory, as their combination with the poles in ϵ coming from counter-terms leads to finite contributions. The procedure described above is only possible if the SI is spontaneously broken (otherwise the perturbative expansion is ill defined), but this is required anyway by phenomenological considerations.

If the construction of quantum scale-invariant theories based on dimensional regularisation is self-consistent, then the result should have a more general character and other renormalisation

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schemes, leading to quantum dilatational invariance, should exist. Lattice regularisation plays a special role in construction and studies of quantum field theories since it allows for non-perturbative approach. The aim of the present Letter is to attempt a non-perturbative lattice construction of *effective* quantum SI theories.

The Letter is organized as follows. In Section 2 we present the idea of lattice regularisation of SI theories with the use of an example of a simple scalar theory. In Section 3 we will discuss the inclusion of gravity. Section 4 is conclusions.

2. Lattice spacing as a dynamical field

To present our main idea, we start with a simple scalar field model, given by the action

$$S = \int d^4x \mathscr{L}, \tag{1}$$

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu \chi)^2 + (\partial_\mu h)^2\right] - V(h, \chi), \tag{2}$$

$$V(h, \chi) = \lambda\left(h^2 - \zeta^2\chi^2\right)^2 + \beta\chi^4. \tag{3}$$

The fields h and χ can be thought of as the Higgs field and the dilaton correspondingly. Here λ and ζ are dimensionless coupling constants. The theory is scale-invariant at the classical level: the action does not change under substitution $h(x) \to \Omega h(\Omega x)$ and $\chi(x) \to \Omega\chi(\Omega x)$, where Ω is a parameter of dilatation transformation. If $\beta > 0$, the vacuum state of the theory $h = \chi = 0$ is scale-invariant. If $\beta < 0$, the theory does not have a ground state. The vacuum state with spontaneously broken SI appears if $\beta = 0$ and corresponds to some point in the flat direction $h^2 - \zeta^2\chi^2 = 0$. The specific values of coupling constants and of the vacuum expectation value of the dilaton χ_0 are not important for what follows.[1]

The scale-invariant perturbative renormalisation of this model has been considered in [8]. It was shown there that if the parameter μ of dimensional regularisation is replaced by an appropriate combination of dynamical fields, then the counter-terms can be chosen in such a way that the divergences are removed and the classical flat direction, existing at $\beta = 0$, is not lifted by quantum corrections. A choice, motivated by cosmological considerations [7], reads

$$\mu^{2\epsilon} \to \left[\omega^2\right]^{\frac{\epsilon}{1-\epsilon}}, \tag{4}$$

where $\omega^2 \equiv (\xi_\chi\chi^2 + \xi_h h^2)$ and ξ_χ, ξ_h are the couplings of the scalar fields to the Ricci scalar, see Section 3. The low energy theory contains a massive Higgs field and the massless dilaton, the latter being a Goldstone boson of the broken scale invariance.[2] A more familiar example of a scale-invariant quantum theory with spontaneous breaking of SI, which appears in standard renormalisation procedure, is $N = 4$ supersymmetric Yang–Mills theory [2].

The requirement that the scale invariance must be spontaneously broken is essential for perturbative construction. Only in this case the perturbative computations can be done. Moreover, if both the ground state and the action respect SI, the resulting theory does not contain any mass scale and thus cannot be accepted phenomenologically.

Let us turn now to lattice regularisation (LR). Consider some space–time lattice of points, with oriented edges connecting them. The space–time is taken to be infinite at the moment. Finite difference scheme can be constructed in many different ways leading to the same result in continuum limit. To simplify the subsequent discussion let us denote by ϕ_{i+} and ϕ_{i-} the values of a generic field ϕ at space–time points corresponding to the i-th edge end. The field difference and the field strength associated to this edge is $\Delta\phi_i = \phi_{i+} - \phi_{i-}$ and $\phi_i = (\phi_{i+} + \phi_{i-})/2$ correspondingly. We assume that the length of the lattice edges a_i may vary in space and time. Then the classical lattice action of the model Eq. (3) becomes

$$S = \sum_i a_i^4\left[\frac{\Delta\chi_i^2 + \Delta h_i^2}{2a_i^2} - V(h_i, \chi_i)\right]. \tag{5}$$

Clearly, this regularisation breaks explicitly the scale invariance. It introduces the lattice spacing a_i, playing the role of the inverse ultraviolet cutoff in quantum theory. Therefore, the only possibility to construct a scale invariant theory at quantum level in the framework of LR, is to promote the lattice spacing to a dynamical field ϕ,

$$a_i^{-1} = \eta\phi_i, \tag{6}$$

where η is some constant, determining how fine is the grid. A possibility to have an ultraviolet cutoff as a dynamical field in SI theories was discussed previously in [11].

The ϕ could be a new field, not present yet in Eq. (3). However, we can assume that our starting action contains already all scalar fields of the theory. Then, in general, ϕ is a function of h and χ. We take it to be $\phi^2 = \omega^2$, in analogy with the previous discussion, see Eq. (4). Then the action on the lattice becomes

$$S = \sum_i\left[\frac{\Delta\chi_i^2 + \Delta h_i^2}{2\omega_i^2\eta^2} - \frac{V(h_i, \chi_i)}{\omega_i^4\eta^4}\right]. \tag{7}$$

It is invariant under the scale transformations

$$h_i \to \Omega h_i, \qquad \chi_i \to \Omega\chi_i. \tag{8}$$

The quantum SI system can be defined by the Euclidean partition function,

$$Z = \prod_i \int \frac{d\chi_i\, dh_i}{\sigma_i^2} e^{-S_E}, \tag{9}$$

where we introduced the scale-invariant path integral measure and made the redefinition of the fields (in such a way that $\omega_i^2 \to \sigma_i^2 = h_i^2 + \chi_i^2$) and coupling constants accounting for renormalisation as follows:

$$S_E = \sum_i\left[\frac{A\Delta\chi_i^2 + B\Delta h_i^2}{2\sigma_i^2} + \frac{C\chi_i^4 + D\chi_i^2 h_i^2 + Eh_i^4}{\sigma_i^4}\right], \tag{10}$$

where A, B, C, D and E are 5 arbitrary parameters. If the theory is considered in a finite space–time volume, the summation in (10) is limited by the total number N of the lattice edges.

The naive continuum limit of the lattice theory (the physical volume is fixed) is achieved by the scaling $\eta \to \kappa\eta$, $(A, B) \to (A, B)/\kappa^2$, $(C, D, E) \to (C, D, E)/\kappa^4$, $N \to \kappa^4 N$, $\kappa \to \infty$, corresponding to a finer covering of the space by the lattice points.

For numerical lattice simulations another choice of variables can be more convenient. For example, the change of variables $\chi = \exp(\theta)\cos(\phi)$, $h = \exp(\theta)\sin(\phi)$ will simplify the kinetic and potential terms.

The phenomenologically interesting theories are those where the scale invariance is spontaneously broken. This does not necessarily happen for all possible choices of parameters in the lattice action (10). To select a class of theories with SI spontaneously broken, one can construct an effective potential $V_{\text{eff}}(\chi, h)$, given by

[1] The choice required by phenomenological considerations corresponds to $\chi_0 \simeq M_{\text{P}}$, where M_{P} is the Planck mass, and $\zeta \sim v/M_{\text{P}} \sim 10^{-16} \lll 1$, where v is the electroweak scale.

[2] In the presence of scale-invariant gravity this particle has only derivative couplings to matter and thus escapes experimental bounds, given the small value of ζ.