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Galois groups in rational conformal field theory II. The discriminant

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Abstract

We express the discriminant of the polynomial relations of the fusion ring, in any conformal field theory, as the product of the rows of the modular matrix to the power -2. The discriminant is shown to be an integer, always, which is a product of primes which divide the level. Detailed formulas for the discriminant are given for all WZW conformal field theories.

The classification of rational conformal field theories is one of the intriguing problems in relation to string theory and to critical phenomena. The study of such theories were initiated in the seminal works [1–4].

Here we continue the work of Ref. [5], where some equations where derived for the fusion ring. Our aim here is to define the discriminant of the relations of the fusion ring. This is expressed as the product of the rows of the modular matrix to the power -2. It is shown to be an integer which has the striking property of being a product of primes dividing the level. We hope that these results will help in the classification of rational conformal field theories.

From the discussion in Ref. [5], we recall the theorem proven originally in Ref. [6], which states as follows:

Theorem 1. *Any fusion ring is a ring of polynomials $Z[x_1, x_2, \dots, x_m]/I$ where I is the ideal of all the polynomials vanishing on the points of the fusion variety, $x^i_\alpha = \frac{S^\dagger_{i\alpha}}{S_{i0}}$, where $\alpha = 1, 2, \dots, m$, label the generators, and i denotes the different points of the variety, whose number is the number of primary fields. Furthermore, the value of any primary field $[a]$ evaluated on any of the points of the fusion variety is given by $p_a(x^1_i, x^2_i, \dots, x^m_i) = \frac{S^\dagger_{a,i}}{S_{0,i}}$.*

For further explanation and a proof see Refs. [5,6].

For simplicity, we assume that the fusion ring is generated by one primary field, which we label as $x = [1]$. It follows that the fusion ring is given by the quotient,

$$R \approx \frac{Z[x]}{(q(x))}, \tag{1}$$

where $Z[x]$ is the ring of polynomials in x with integer coefficients, and $q(x)$ is the relation in this ring. We can write the primary field in this ring as $p_r(x) = x^r + a_1 x^{r-1} + \cdots$, where by convention $p_0(x) = 1$ and $p_1(x) = x$ and r goes from 0 to $n-1$. Using

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the theorem we may write the following matrix equation,

$$M_{i\alpha}=\frac{S^\dagger_{i\alpha}}{S_{i0}}=\begin{pmatrix}1 & 1 & \dots & 1\\ x_0 & x_1 & \dots & x_{n-1}\\ p_2(x_0) & p_2(x_1) & \dots & p_2(x_{n-1})\\ \dots & \dots & \dots & \dots\\ p_{n-1}(x_0) & p_{n-1}(x_1) & \dots & p_{n-1}(x_{n-1})\end{pmatrix}, \tag{2}$$

where $x_0, x_1, \dots, x_{n-1}$ label the points of the fusion variety, i.e., the solutions of the equation $q(x)=0$.

Now, take the determinant of both sides of Eq. (2). On the right side, since we can eliminate the lower order terms by adding rows recursively, we have the Vandermonde determinant,

$$\det(M)=\det_{i,j} x_i^j. \tag{3}$$

As is well known, this determinant is given as the product,

$$\det(M)=\prod_{i<j}(x_i-x_j). \tag{4}$$

Squaring the Vandermonde determinant we get the discriminant,

$$D=\prod_{i<j}(x_i-x_j)^2, \tag{5}$$

which since it is a symmetric polynomial in x_i can be expressed in terms of the coefficients of the polynomial $q(x)$. Say,

$$q(x)=x^m-S_1x^{m-1}+S_2x^{m-2}-\dots+(-1)^mS_m, \tag{6}$$

where the S_r are the symmetric polynomials of the roots,

$$S_r=\sum_{i_1<i_2<\dots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r}. \tag{7}$$

The discriminant may easily be computed for any equation. For example, see the appendix of Ref. [6] for a Mathematica program. We quote the result for equations of order 2 and 3,

$$R_2=S_1^2-4S_2,\qquad R_3=S_1^2S_2^2-4S_2^3-4S_1^3S_3+18S_1S_2S_3-27S_3^2. \tag{8}$$

Very importantly, since the coefficients S_i are integers, the discriminant is always an integer.

Now, consider the left-hand side of Eq. (2). Since S obeys $S^2=C$, where C is the charge conjugation matrix, $\det(S)=i^s$, where s is some integer. So we find for the determinant,

$$\det(M)=i^s\prod_i S_{i0}^{-1}. \tag{9}$$

Comparing Eqs. (4), (9), we find immediately an expression for the discriminant,

$$|D|=\prod_i S_{i0}^{-2}. \tag{10}$$

Since we are free to act with an element of the Galois group, G, we find also that

$$|D|=\prod_i S_{i,a}^{-2}, \tag{11}$$

where a is any primary field in the path of the Galois element,

$$a=t(0),\quad \text{where } t\in G. \tag{12}$$

Quite strikingly, since the discriminant is an integer, we find that the product of the elements of the modular matrix is always the inverse square root of an integer,

$$D=\prod_i S_{0,i}^{-2}=\text{integer}. \tag{13}$$

Although our proof utilized the assumption of one generator, we believe that Eq. (13) holds for any number of generators since it depends only on the product of the rows of the modular matrix. In particular, since the generator is not used, the discriminant is independent of the choice of generators, and is always the same for any generator.