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Isotropic coordinates for Schwarzschild black hole radiation

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Abstract

The isotropic coordinate system of Schwarzschild spacetime has several attractive properties similar with the Painlevé–Gullstrand coordinates. The purpose for us to choose the isotropic coordinates is to resolve the ambiguities of the tunneling picture in Hawking radiation. Based on energy conservation, we investigate Hawking radiation as massless particles tunneling across the event horizon of the Schwarzschild black hole in the isotropic coordinates. Because the amplitude for a black hole to emit particles is related to the amplitude for it to absorb, we must take into account the contribution of ingoing solution to the action, $\operatorname{Im} S = \operatorname{Im} S_{\text{out}} - \operatorname{Im} S_{\text{in}}$. It will be shown that the imaginary part of action for ingoing particles is zero ($\operatorname{Im} S_{\text{in}} = 0$) in the Painlevé–Gullstrand coordinates, so the equation $\operatorname{Im} S = \operatorname{Im} S_{\text{out}} - \operatorname{Im} S_{\text{in}}$ is valid in both the isotropic coordinates and the Painlevé–Gullstrand coordinates.

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1. Introduction

It is known that nothing can escape from a black hole at classical level, but particles can tunnel though the horizon if we take into account the effect of quantum mechanics. By calculating the response of quantum field to collapse geometry, Hawking radiation was derived: black holes radiate thermally with the temperature $T = \frac{\kappa}{2\pi}$, κ is the surface gravity of black holes [1]. Recently Parikh and Wilczek provide a short, direct semi-classical derivation of black hole radiance [2–4]. A new method was introduced to calculate the emission rate in the Painlevé–Gullstrand coordinate system well-behaved at the horizon. Based on energy conservation and the self-gravitational interaction of the radiation, it was shown that the radiation spectrum can not be strictly thermal and the result is consistent with an underlying unitary theory. Many authors also study this problems in dynamics background and have done excellent works [5–13]. In this Letter, we attempt to extend this method to Schwarzschild black holes in isotropic coordinates. It was pointed out in [14] that there are closely correlation between the amplitude for a black hole to emit particles and the amplitude for it to absorb. With this idea, we study the Hawking radiation in Schwarzschild black holes in dynamical background.

2. Tunneling

To describe the tunneling picture, first we need to find a coordinate well-behaved at the event horizon. In this Letter, we choose the isotropic coordinates to describe across-horizon phenomena. The isotropic coordinates have several attractive properties similar with the Painlevé–Gullstrand coordinates: There are non-singular at the horizon, the time direction is a Killing vector and the isotropic coordinates satisfy Landau's condition of the coordinate clock synchronization

$$\frac{\partial}{\partial x_j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\partial}{\partial x_i}\left(-\frac{g_{0j}}{g_{00}}\right) \quad (i, j = 1, 2, 3). \tag{1}$$

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The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\,d\Omega^2. \tag{2}$$

The isotropic coordinates are obtained from the Schwarzschild coordinates via the transformation [15,16]

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2, \tag{3}$$

then the isotropic coordinates are derived

$$ds^2 = -\left(\frac{\rho - \frac{M}{2}}{\rho + \frac{M}{2}}\right)^2 dt^2 + \left(\frac{\rho + \frac{M}{2}}{\rho}\right)^4 \left(d\rho^2 + \rho^2\,d\Omega^2\right). \tag{4}$$

We can see there is non-singular at the horizon $r = 2M$, where M is the mass of black hole. The property of the spacetime is static. On the assumption that $d\theta = d\varphi = 0$, the radial null geodesics of massless particles are given by

$$\dot\rho \equiv \frac{d\rho}{dt} = \pm\frac{(\rho - \frac{M}{2})\rho^2}{(\rho + \frac{M}{2})^3}, \tag{5}$$

where the upper (lower) sign in Eq. (5) corresponds to outgoing (ingoing) geodesics, under the implicit assumption that time t increases towards the future. In tunneling picture, a pair of virtual particles is spontaneously created just inside (outside) the horizon. The positive energy virtual particle can tunnel through the horizon and materialize as a real particle to escape to infinity, while the negative energy particle is absorbed by the black hole. Based on energy conservation, the mass of the black hole must go down as it radiates. If we fix the total mass and allow the hole mass to fluctuate and consider the particle as an sphere shell of energy ω, the line element can be obtained by considering the self-gravity effect [17]

$$ds^2 = -\left(\frac{\rho - \frac{(M-\omega)}{2}}{\rho + \frac{(M-\omega)}{2}}\right)^2 dt^2 + \left(\frac{\rho + \frac{(M-\omega)}{2}}{\rho}\right)^4 \left(d\rho^2 + \rho^2\,d\Omega^2\right), \tag{6}$$

and Eq. (5) becomes

$$\dot\rho \equiv \frac{d\rho}{dt} = \pm\frac{(\rho - \frac{(M-\omega)}{2})\rho^2}{(\rho + \frac{(M-\omega)}{2})^3}. \tag{7}$$

Based on Eq. (3), the equation $\rho = \frac{M}{2}$ corresponds to $r = 2M$, we can see the event horizon and the infinite red-shift surface coincide with each other in the isotropic coordinates so that the semi-classical WKB approximation can be used, then the tunneling probability can be expressed as

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S}. \tag{8}$$

In [14], the Feynman path-integral was used to derive the thermal radiation emitted by black holes. It was shown that the ratio of emission and absorption probabilities for energy E is

$$P_{\text{emission}} = \exp\left(-\frac{E}{T_H}\right)P_{\text{absorption}}. \tag{9}$$

Based on Eqs. (8) and (9), we obtain

$$\Gamma \sim \exp\bigl(-2(\mathrm{Im}\,S_{\text{out}} - \mathrm{Im}\,S_{\text{in}})\bigr) = \exp\left(-\frac{E}{T_H}\right), \tag{10}$$

then we can see the $\mathrm{Im}\,S$ are composed of two parts

$$\mathrm{Im}\,S = \mathrm{Im}\,S_{\text{out}} - \mathrm{Im}\,S_{\text{in}}. \tag{11}$$

The imaginary part of the action for the outgoing particles can be written as

$$\mathrm{Im}\,S_{\text{out}} = \mathrm{Im}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}} p_\rho\,d\rho = \mathrm{Im}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}}\int_0^{p_\rho} dp'_\rho\,d\rho, \tag{12}$$

where $\rho_{\text{in}} = \frac{M}{2}$ and $\rho_{\text{out}} = \frac{M-\omega}{2}$. Based on the Hamilton's equation $\dot\rho = \frac{dH}{dp_\rho}|_\rho = \frac{d(M-\omega)}{dp_\rho}$, then Eq. (12) becomes

$$\mathrm{Im}\,S_{\text{out}} = \int_M^{M-\omega}\int_{\frac{M}{2}}^{\frac{M-\omega}{2}} \frac{d\rho}{\dot\rho}\,dH. \tag{13}$$

Inserting Eq. (7) into Eq. (13), we obtain

$$\mathrm{Im}\,S_{\text{out}} = \int_0^{\omega}\int_{\frac{M}{2}}^{\frac{M-\omega}{2}} \frac{(\rho + \frac{(M-\omega')}{2})^3}{(\rho - \frac{(M-\omega')}{2})\rho^2}\,d\rho\,d(-\omega'). \tag{14}$$

It is obvious that $\rho = \frac{M-\omega'}{2}$ is a single pole in Eq. (14), then we can evaluate the integral by deforming the contour around the pole. We must pay attention to a subtle point which was pointed out in [18] that when one deforms the contour based on Eq. (3), the semi-circular contour in Eq. (14) gets transformed into a quarter circle so that one gets $i\frac{\pi}{2}$ Residue rather than $i\pi$ Residue. In detail, we make the change of variables $r - 2M = \epsilon e^{i\theta}$ and insert it into Eq. (3), then

$$\rho + M + \frac{M^2}{4\rho} = 2M + \epsilon e^{i\theta}. \tag{15}$$

The solution is

$$\rho = \frac{1}{2}\bigl(M + \epsilon e^{i\theta} \pm \bigl(2M + \epsilon e^{i\theta}\bigr)^{\frac{1}{2}}\sqrt{\epsilon}\,e^{\frac{i\theta}{2}}\bigr). \tag{16}$$

In the limit $\epsilon \to 0$, Eq. (16) becomes $\rho - \frac{M}{2} = \pm\sqrt{2M\epsilon}\,e^{\frac{i\theta}{2}}$. Now it is obvious that the semi-circle contour gets transformed into a quarter circle, then we obtain

$$\mathrm{Im}\,S_{\text{out}} = \int_0^{\omega} 2\pi(M - \omega')\,d\omega' = 2\pi\omega\left(M - \frac{\omega}{2}\right). \tag{17}$$

In the same way, we can obtain the imaginary part of action for the ingoing particles

$$\mathrm{Im}\,S_{\text{in}} = -\int_0^{\omega}\int_{\frac{M}{2}}^{\frac{M-\omega}{2}} \frac{(\rho + \frac{(M-\omega')}{2})^3}{(\rho - \frac{(M-\omega')}{2})\rho^2}\,d\rho\,d(-\omega') = -2\pi\omega\left(M - \frac{\omega}{2}\right). \tag{18}$$