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Noninvariant zeta-function regularization in quantum Liouville theory ☆

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Abstract

We consider two possible zeta-function regularization schemes of quantum Liouville theory. One refers to the Laplace–Beltrami operator covariant under conformal transformations, the other to the naive noninvariant operator. The first produces an invariant regularization which however does not give rise to a theory invariant under the full conformal group. The other is equivalent to the regularization proposed by A.B. Zamolodchikov and Al.B. Zamolodchikov and gives rise to a theory invariant under the full conformal group.

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Quantum Liouville theory has been the subject of intense study following different lines of attack. While the bootstrap [1–5] starts from the requirement of obtaining a theory invariant under the full infinite dimensional conformal group, the more conventional field theory techniques like the Hamiltonian and the functional approaches depend in a critical way on the regularization scheme adopted. In the hamiltonian treatment [6] for the theory compactified on a circle the normal ordering regularization gives rise to a theory invariant under the full infinite dimensional conformal group.

It came somewhat of a surprise that in the functional approach the regularization which realizes the full conformal invariance is the noninvariant regularization introduced by A.B. Zamolodchikov and Al.B. Zamolodchikov (ZZ) [4]. In [7] it was shown that such a regularization provides the correct quantum dimensions to the vertex functions on the sphere at least to two loops while in [8] it was shown that such a result holds true to all order perturbation theory on the pseudosphere. Here we consider the approach in which the determinant of a noncovariant operator is computed in the framework of the zeta function regularization and show that this procedure is equivalent to the noninvariant regularization of the Green function at coincident points proposed by ZZ [4], and extensively used in [7–10]. For definiteness we shall refer to the case of sphere topology.

The complete action is given by $S_L[\varphi_B,\chi] = S_{\rm cl}[\varphi_B] + S_q[\varphi_B,\chi]$ where [7]

$$S_{\rm cl}[\varphi_B] = \lim_{\substack{\varepsilon_n\to 0\\ R\to\infty}} \frac{1}{b^2}\left[\frac{1}{8\pi}\int_\Gamma \left(\frac{1}{2}(\partial_a\varphi_B)^2 + 8\pi\mu b^2 e^{\varphi_B}\right)d^2z \right.$$
$$\left. - \sum_{n=1}^N \left(\eta_n \frac{1}{4\pi i}\oint_{\partial\Gamma_n} \varphi_B\left(\frac{dz}{z-z_n} - \frac{d\bar z}{\bar z - \bar z_n}\right) + \eta_n^2 \log\varepsilon_n^2\right) + \frac{1}{4\pi i}\oint_{\partial\Gamma_R}\varphi_B\left(\frac{dz}{z} - \frac{d\bar z}{\bar z}\right) + \log R^2\right] \tag{1}$$

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and

$$S_q[\varphi_B,\chi]=\lim_{\substack{\varepsilon_n\to 0\\ R\to\infty}}\left[\frac{1}{4\pi}\int_\Gamma\big((\partial_a\chi)^2+4\pi\mu e^{\varphi_B}\big(e^{2b\chi}-1-2b\chi\big)\big)\,d^2z\right.$$
$$\left.+\big(2+b^2\big)\log R^2+\frac{1}{4\pi i}\oint_{\partial\Gamma_R}\varphi_B\left(\frac{dz}{z}-\frac{d\bar z}{\bar z}\right)+\frac{b}{2\pi i}\oint_{\partial\Gamma_R}\chi\left(\frac{dz}{z}-\frac{d\bar z}{\bar z}\right)\right]. \tag{2}$$

In Eqs. (1), (2) Γ is a disk of radius R from which disks of radius ε_n around the singularities have been removed.

We recall that $S_{\rm cl}$ is $O(1/b^2)$ while the first integral appearing in the quantum action (2) can be expanded as

$$\frac{1}{4\pi}\int_\Gamma\left((\partial_a\chi)^2+8\pi\mu b^2e^{\varphi_B}\chi^2+8\pi\mu b^2e^{\varphi_B}\left(\frac{4b\chi^3}{3!}+\frac{8b^2\chi^4}{4!}+\cdots\right)\right)d^2z. \tag{3}$$

The quantum n-point function is given by

$$\big\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\cdots V_{\alpha_n}(z_n)\big\rangle=e^{-S_{\rm cl}[\varphi_B]}\int\mathcal{D}[\chi]e^{-S_q}, \tag{4}$$

where the φ_B appearing in the classical and quantum actions is the solution of the classical Liouville equation in presence of n sources

$$-\Delta\varphi_B+8\pi\mu b^2e^{\varphi_B(z)}=8\pi\sum_{i=1}^n\eta_i\delta^2(z-z_i), \tag{5}$$

where $\eta_i=b\alpha_i$ and the vertex functions are given by

$$V_\alpha(z)=e^{2\alpha\phi(z)}=e^{\eta\varphi(z)/b^2};\qquad \varphi=2b\phi=\varphi_B+2b\chi. \tag{6}$$

We recall that the action (1) ascribes to the vertex function $V_\alpha(z)$ the semiclassical dimension $\Delta_{\rm sc}(\alpha)=\alpha(1/b-\alpha)$ [11].

In performing the perturbative expansion in b we have to keep $\eta_1,\dots,\eta_n$ constant [3]. The one loop contribution to the n-point function is given by

$$K^{-\frac12}=\int\mathcal{D}[\chi]e^{-\frac12\int\chi(z)D\chi(z)\,d^2z}, \tag{7}$$

where

$$D=-\frac{2}{\pi}\partial_z\partial_{\bar z}+4\mu b^2e^{\varphi_B}\equiv-\frac{1}{2\pi}\Delta+m^2e^{\varphi_B}. \tag{8}$$

The usual invariant zeta-function technique [12] for the computation of the functional determinant K consists in writing

$$\int\chi(z)D\chi(z)\,d^2z=\int\chi(z)\left(-\frac{1}{2\pi}\Delta_{\rm LB}+m^2\right)\chi(z)\,d\rho(z) \tag{9}$$

being $d\rho(z)=e^{\varphi_B(z)}\,d^2z$ the conformal invariant measure and

$$\Delta_{\rm LB}=e^{-\varphi_B}\Delta \tag{10}$$

the covariant Laplace–Beltrami operator on the background $\varphi_B(z)$ generated by the n charges. The determinant of the elliptic operator $-\frac{1}{2\pi}\Delta_{\rm LB}+m^2$ is defined through the zeta-function

$$\zeta(s)=\sum_{i=1}^\infty\lambda_i^{-s} \tag{11}$$

being λ_i the eigenvalues of the operator H,

$$H\varphi_i=\lambda_i\varphi_i\quad\text{where } H=-\frac{1}{2\pi}\Delta_{\rm LB}+m^2. \tag{12}$$

For an elliptic operator the sum (11) converges for $\operatorname{Re}s$ sufficiently large and positive and the determinant is defined by analytic continuation as

$$-\log(\operatorname{Det}H)=\zeta'(0). \tag{13}$$