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Statistical entropy of three-dimensional q-deformed Kerr–de Sitter space

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Abstract

A quantum deformation of three-dimensional de Sitter space was proposed in hep-th/0407188. We use this to calculate the entropy of Kerr–de Sitter space, using a canonical ensemble, and find agreement with the semiclassical result.

Keywords: de Sitter; Conformal field theory; Quantum group

1. Introduction

Black holes are known to carry entropy proportional to their horizon area [1,2]. One of the main successes of string theory has been to provide a microscopic interpretation of this entropy [3], at least in certain cases. Since the semiclassical arguments for this horizon area are not specific to horizons surrounding black holes, they should also apply to cosmological horizons [4–6]. It would be of great interest to have microscopic state-counting arguments for such situations.

The natural place to investigate cosmological horizons is de Sitter space, the maximally symmetric spacetime with constant positive cosmological constant Λ [7–11]. Current observations suggest that our universe is now Λ-dominated, and thus asymptotically de Sitter in the future [12–14].

Following the great success of the AdS/CFT correspondence [15], there have been suggestions of a dual conformal field theory, living on the conformal boundary of de Sitter [8,16]. Unlike the anti-de Sitter case, this boundary is spacelike, and time-translation in the bulk corresponding to scale transformation on the boundary [17–19]. It also has two disconnected components, and it is not clear whether the boundary theory should live on one or both [7].

Since the area of an observer's cosmological horizon is finite, de Sitter has finite Bekenstein–Hawking entropy. This immediately causes problems with finding a state counting interpretation, since the isometry group is non-compact [20] and hence only has infinite-dimensional unitary representations. This apparent contradiction is even stronger if the dimension of the Hilbert space is also finite [21–23]. It has been suggested that the correct inner product is not the naive local one, which changes the notion of unitarity [24,25]. (Another approach is given in [26,27].)

Some of these difficulties might be tamed by noncommutative geometry [28,29]. The approach used in [30] is to deform the group of isometries into a quantum group. This was further studied in [31–33] and is the approach followed here. Quantum deformations of the Lorentz group in various dimensions are studied in [34,35] and in [36,37].

The plan of the Letter is as follows. Section 2 of this Letter is mostly a recap of [30,31]. Section 3 contains a novel calculation of the entropy. The interpretation of the result is discussed and concluding remarks made in Section 4.

2. q-deformed de Sitter space

Three-dimensional de Sitter space can be defined as the hyperboloid $-(x^0)^2 + \sum_{i=1}^{3}(x^i)^2 = \ell^2$ in Minkowski space. This is a spacetime of constant curvature, representing a vacuum with positive cosmological constant $\Lambda = 1/\ell^2$.

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The isometries of this hyperboloid are just the rotations and boosts of the embedding space, which generate the Lorentz group $SO(3,1)$. We will focus on the Lie algebra, rather than the global properties of the group. The complex combinations of generators $X_i = J_i + iK_i$ (left) and $\bar{X}_i = J_i - iK_i$ (right) each obey the $su(2)$ commutation relation $[X_i, X_j] = i\epsilon_{ijk}X_k$, and commute with each other.

In the complex algebra, the fact that J and K are Hermitian is encoded in the star operation $J^\star = J$, $K^\star = K$, and so $X_i^\star = \bar{X}_i$. The use of this star-structure specifies that we are dealing with the non-compact real form $so(3,1)$. We will also use the basis given by:

$$\begin{aligned} L_0 &= X_1, & \bar{L}_0 &= -\bar{X}_1, \\ L_1 &= X_2 - iX_3, & \bar{L}_1 &= -\bar{X}_2 - i\bar{X}_3, \\ L_{-1} &= -X_2 - iX_3, & \bar{L}_{-1} &= \bar{X}_2 - i\bar{X}_3. \end{aligned}$$

These generators form the $n = 0, \pm 1$ part of the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$, but with real form

$$L_n^\star = -\bar{L}_n. \tag{1}$$

A field in de Sitter space will transform under isometries in some representation of this algebra. Since the group is non-compact, there are no finite-dimensional unitary representations, thus any field has infinitely many modes. For a field of mass $m > \ell$ the representation is in the principal series [38–40].

It was proposed in [30] that the Lie algebra of isometries should be deformed to a quantum group (Hopf algebra) [41, 42]. Taking the deformation parameter to be a root of unity

$$q = e^{2\pi i/N}$$

limits the dimension of an irreducible representation to at most N. In particular, the deformed versions of non-compact algebras can have finite-dimensional unitary representations, which become infinite in the classical limit $q \to 1$ [43,44]. This was done explicitly for dS_2's $so(2,1)$ principal series in [30]. The relation between N and gravity quantities will be fixed momentarily.

In dS_3 however there is a complication which does not arise in dS_2: even the deformed algebra cannot have non-trivial unitary representations [31]. Suppose $|\psi\rangle$ is an eigenstate of L_0 and $\bar{L}_0$ in a unitary representation. Then the state $L_{\pm 1}|\psi\rangle$ has zero norm, since $L_1^\star$ does not lower the eigenvalue L_1 raised. So the representation must be trivial. (In the infinite-dimensional principal series representation, such a $|\psi\rangle$ lies outside the Hilbert space.)

Similar problems with unitarity are found in [37] in attempting to deform this and higher Lorentz groups, and multi-parameter families of deformations were studied in [36].

These algebraic problems are related to the problem of defining an inner product for fields on de Sitter space, which in turn induces a particular adjoint. The standard local Klein–Gordon one induces (1). Witten proposed to use the path integral from asymptotic past to future, with an extra insertion of CPT [24]. Choosing the parity operation to be $Px^3 = -x^3$, [31] showed that this induces

$$L_n^\dagger = -L_n, \qquad \bar{L}_n^\dagger = -\bar{L}_n \tag{2}$$

or $X_{1,2}^\dagger = -X_{1,2}$, $X_3^\dagger = X_3$ and the same on the right. This amounts to using the (non-compact) split real form $su(1,1) \oplus su(1,1)$, instead of $so(3,1)$.[1]

With this real form, the natural deformation of the algebra to use is

$$U_q\big(su(1,1)\big) \oplus U_q\big(su(1,1)\big).$$

The quantum group $U_q(su(1,1))$ has unitary representations of dimension N. These are representations without highest weight, having $(X_\pm)^N \neq 0$, and are called cyclic representations ($\mathcal{B}$ in [41]). It was shown in [30] that the parameters of a cyclic representation can be chosen so as to give the same Casimirs as the classical $su(1,1) = so(2,1)$ principal series, and in [31] that a left–right product of two cyclic representations has the correct Casimirs to match the $so(3,1)$ principal series.

The geodesics lying in the embedding space's 0–1 plane are the north and south poles of de Sitter space. The south pole is $r = 0$ in the static coordinate patch, whose metric is

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - r^2/\ell^2} + r^2\,d\phi^2. \tag{3}$$

The generator of time translations here is

$$-i\partial_t = K_1 = -i(L_0 + \bar{L}_0).$$

At the antipodal point $-x^\mu$ this generates instead reverse time translation. (This is the standard situation for a thermofield double, the canonical example of which is Rindler space.)

In these coordinates the horizon is at $r = \ell$. It has Hawking temperature $T = 1/2\pi$ which can be derived most transparently for our purposes by tracing over modes living behind the horizon (which have negative frequency) to produce southern density matrix [25]

$$\rho^{\text{south}} \propto e^{-\beta K_1}.$$

In the classical (principal series) case this operator has a continuous spectrum, while a single irreducible cyclic representation of the quantum group has eigenvalues spaced approximately $1/\ell$ apart. So it was proposed in [31] that the appropriate quantum representations are not the cyclic representations $\mathcal{B}$, of dimension N, but rather reducible representations $\bigoplus_{i=1}^N \mathcal{B}_i$ of dimension N^2. There is one phase parameter of the cyclic representation not fixed by matching the principal series's Casimirs, and the sum is over different choices of this phase. In the resulting twisted representation, $-i(L_0 + \bar{L}_0)$ has eigenvalues spaced $\sim 1/N\ell$, thus tending to a continuum in the classical limit.

The natural choice for N is the de Sitter radius in Planck units: we set

$$N = \frac{\ell}{G}.$$

The maximum eigenvalue of J_1, the generator of rotations about the south pole, is of order N, so this can be viewed as allowing

[1] Throughout this Letter we use $\star$ for the $so(3,1)$ involution (1), and $\dagger$ for this one.