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# Heavy-quark QCD vacuum polarisation function: analytical results at four loops

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**Abstract**

The first two moments of the heavy-quark vacuum polarisation function at four loops in quantum chromo-dynamics are found in fully analytical form by evaluating the missing massive four-loop tadpole master integrals.


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## 1. Introduction

During the past several years, there has been significant progress in the determination of the values of the strong-coupling constant $\alpha_s$ at the $Z$-boson mass scale and the heavy-quark masses $m_h$ in perturbative quantum chromo-dynamics (QCD). The matching relations which determine the finite discontinuities of $\alpha_s$ at the heavy-quark thresholds in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme and are necessary for a precise determination of $\alpha_s$ from a global data fit were recently presented in semi-analytical [1] and analytical [2] form at the four-loop level. Furthermore, there is significant progress in multi-loop technology aiming towards an analytical evaluation of the QCD $\beta$ function at five loops.

In order to precisely determine heavy-quark masses with the help of QCD sum rules [3], it is necessary to know in detail the heavy-quark contribution to the QCD vacuum polarisation function. The most important ingredients for this analysis are the lowest moments of the polarisation function (see Ref. [4]). Indeed, it was found that the first moment is best suited for this analysis, since it has the weakest dependence on non-perturbative effects and the details of the threshold region, thus leading to the smallest theoretical uncertainty [5].

At the four-loop level, the study of the first two moments has been started in Ref. [6] with calculations of diagrams including two internal loops of massive and massless quarks coupled to gluons. The results in Ref. [6] contain two numerical constants, denoted as $N_{10}$ and $N_{20}$, which were not expressed in terms of basic transcendental numbers. In the meantime, the corresponding Feynman integrals have been calculated analytically [7,8]. The terms proportional to $\alpha_s^j n_l^{j-1}$, where $n_l$ in the number of light quarks, are known in all orders $j$ of perturbative QCD [9]. Refs. [10,11] contain the complete four-loop contributions to the first two Taylor coefficients from non-singlet diagrams. The singlet contributions were studied in Refs. [11,12].

The results of Refs. [10,11] contain some four-loop tadpoles which are only given in numerical form. The purpose of this Letter is to evaluate them in terms of standard transcendental numbers, so as to represent the first two moments of the heavy-quark vacuum polarisation function at four loops in completely analytical form.

The content of this Letter is as follows. Section 2 contains the basic formulae. In Section 3, we present our analytical re-

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sults for the tadpoles that, in Refs. [10,13], are called $T_{54}$, $T_{62}$, and $T_{91}$ as well as for the first two moments of the heavy-quark vacuum polarisation at four loops in QCD. A brief summary is given in Section 4.

## 2. Approach

Since the vector current $j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)$ constructed from the heavy-quark field $\Psi(x)$ is conserved, the correlator,

$$\Pi^{\mu\nu}(q) = i\int dx\, e^{iqx}\langle 0|Tj^\mu(x)j^\nu(0)|0\rangle = \left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\Pi\left(q^2\right), \tag{1}$$

can be expressed in terms of a single scalar function $\Pi(q^2)$. The latter is of great phenomenological interest because it is related to the experimental observable

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi\, \mathrm{Im}\, \Pi(s + i\epsilon), \tag{2}$$

where $s$ is the square of the $e^+e^-$ centre-of-mass energy. Eq. (2) is equivalent to an infinite number of equalities between experimental moments,

$$M_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R(s), \tag{3}$$

and their theoretical counterparts

$$M_n^{\text{th}} = \frac{9e_h^2}{4}\left(\frac{1}{4\bar{m}_h^2}\right)^n \bar{C}_n, \tag{4}$$

where $e_h$ is the electric-charge quantum number of the heavy quark, $\bar{m}_h = m_h(\bar{m}_h)$, and $\bar{C}_n$ are the coefficients of the Taylor expansion

$$\Pi\left(q^2\right) = \frac{3e_h^2}{16\pi^2}\sum_{n\geqslant 0}\bar{C}_n\bar{z}_n \tag{5}$$

in $\bar{z} = q^2/(4\bar{m}_h^2)$. The perturbative calculation of

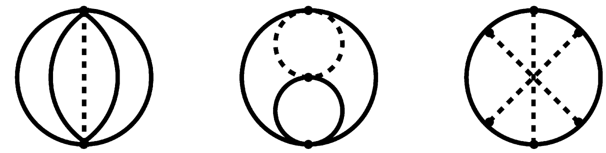
$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3\bar{C}_n^{(3)} + \cdots, \tag{6}$$

where $\alpha_s = \alpha_s(\bar{m}_h^2)$, only involves tadpole diagrams and is, therefore, considerably simpler than the one of $\Pi(q^2)$, which has a complicated dependence on $\bar{z}$. Here, it is understood that $\alpha_s(\mu)$ and $m_h(\mu)$ are defined in the $\overline{\text{MS}}$ renormalisation scheme, which is based on dimensional regularisation with $D = 4 - 2\epsilon$ space–time dimensions and 't Hooft mass scale $\mu$.

Thanks to the strong hierarchy among the quark masses, one may split the number $n_f = n_l + n_h$ of quark flavours into $n_h = 1$ massive ones and $n_l$ massless ones. In the following, we keep the variable $n_h$ generic.

## 3. Results

In our previous paper [8], we introduced a technique that allows one to analytically evaluate a large class of four-loop tadpole diagrams with one non-zero mass. Specifically, the considered diagrams are transformed into integral representations whose integrands contain only one-loop tadpoles with new propagators having masses that depend on the variables of integration. This technique is based on the differential equation method [14]. A similar technique was applied to certain types of two-loop diagrams in Ref. [15].



$T_{54}$ $T_{62}$ $T_{91}$

Fig. 1. Massive four-loop tadpole diagrams $T_{54}$, $T_{62}$, and $T_{91}$.

To illustrate the usefulness of this technique, in Ref. [8], we also evaluated the tadpoles that are denoted in Ref. [6] as $N_{10}$ and $N_{20}$. The finite parts of our results agree with the numerical results of Ref. [6] and the analytical ones of Ref. [7], while our analytical results for the $O(\epsilon)$ terms are new.

These results together with those from Ref. [2] offer us the opportunity to obtain analytical results for terms of the expansion in $\epsilon$ of the master integrals denoted as $T_{54}$, $T_{62}$, and $T_{91}$ in recent papers [10,13] for which only numerical values have been available so far. The corresponding Feynman diagrams are depicted in Fig. 1. Specifically, $T_{62}$ can be extracted from $N_{10}$ [8] using Eqs. (16) and (18) of Ref. [6]. This result can then be used in combination with $T_{91}$ [2] to extract $T_{54}$ from Eq. (50) of Ref. [13]. The results read

$$\begin{aligned}\frac{T_{54}}{N} = {}& -\frac{1}{\epsilon^4} - \frac{9}{2\epsilon^3} - \frac{415}{36\epsilon^2} - \frac{4991}{216\epsilon} - \frac{89383}{1296} + \epsilon\left(-\frac{3367679}{7776}\right.\\ & \left.+ \frac{1792}{9}\zeta(3)\right) + \epsilon^2\left(-\frac{137735095}{46656} - \frac{4352\pi^4}{135}\right.\\ & \left.+ \frac{68992}{27}\zeta(3) + \frac{1024}{3}b_4\right) + \epsilon^3\left(-\frac{4908181487}{279936}\right.\\ & - \frac{167552\pi^4}{405} + \frac{17408\pi^4}{45}\ln 2 + \frac{1602496}{81}\zeta(3)\\ & \left.- \frac{87296}{3}\zeta(5) + \frac{39424}{9}b_4 + 4096b_5\right) + O\left(\epsilon^4\right),\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{T_{62}}{N} = {}& \frac{2}{3\epsilon^4} + \frac{4}{\epsilon^3} + \frac{38}{3\epsilon^2} + \frac{1}{\epsilon}\left(\frac{44}{3} + \frac{16}{3}\zeta(3)\right) - 118 - \frac{4\pi^4}{15}\\ & + 88\zeta(3) + \epsilon\left(-1156 - \frac{374\pi^4}{45} + \frac{2152}{3}\zeta(3)\right.\\ & \left.+ 96\zeta(5) + 64b_4\right) + \epsilon^2\left(-\frac{20938}{3} - \frac{710\pi^4}{9}\right.\\ & + \frac{2416\pi^4}{45}\ln 2 - \frac{8\pi^6}{21} + \frac{12952}{3}\zeta(3) + \frac{64}{3}\zeta^2(3)\\ & \left.- 2400\zeta(5) + 704b_4 + 512b_5\right) + O\left(\epsilon^3\right),\end{aligned} \tag{8}$$

$$T_{91} = -\frac{53\pi^4}{15}\ln 2 + \frac{873}{2}\zeta(5) - 48b_5 + O(\epsilon), \tag{9}$$