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PHYSICS LETTERS B

Physics Letters B 635 (2006) 131–135

www.elsevier.com/locate/physletb

M-theory and the string genus expansion

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Received 2 February 2006; accepted 12 February 2006

Available online 23 February 2006

Editor: L. Alvarez-Gaumé

Abstract

The partition function of the membrane is investigated. In particular, the case relevant to perturbative string theory of a membrane with topology $S^1 \times \Sigma$ is examined. The coupling between the string world sheet Euler character and the dilaton is shown to arise from a careful treatment of the membrane partition function measure. This demonstrates that the M-theory origin of the dilaton coupling to the string world sheet is quantum in nature.

1. Introduction

The basis for string perturbation theory is the coupling of the world sheet Euler character, χ, to the dilaton, Φ [1]. It is this coupling that determines the relationship between the topology of the string world sheet and the string coupling constant and allows string theory at small coupling to be organised in terms of a genus expansion. To see this, consider how the string coupling constant g_s controls the amplitude for a closed string to split into two. This is the so-called "trousers" diagram. To increase the genus of any amplitude by one results in an extra multiplicative factor of g_s^2 for that amplitude. An amplitude corresponding to a diagram with e external legs and l loops, will therefore contain a factor of g_s^{e-2+2l}. Orientable two-dimensional surfaces are classified by their genus g or Euler character χ together with the number of ends (corresponding to external legs) where $\chi = 2(1-g)$ and $g = l + e/2$. Thus each amplitude is weighted by a factor of $g_s^{-\chi}$.

Despite the well-known connection between membranes, fundamental strings and D-branes [2–5],[1] the M-theory origin of the string world sheet Euler character coupling to the dilaton has been regarded as something of a puzzle. To appreciate why the M-theory interpretation for this coupling has so far remained mysterious, let us make a few observations that will be important in understanding the M-theory origin of this term. In what follows, we will consider only string world sheets without boundaries i.e. vacuum diagrams; the extension to include boundaries is essentially trivial. Type IIA string theory is obtained from M-theory by compactification on a Kałuża–Klein circle of radius R_{11}. In string theory, the coupling of the dilaton to the world sheet Σ is given by a contribution to the string action of

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{\tilde{\gamma}}\,R^{(2)}\Phi, \tag{1}$$

where $R^{(2)}$ is the Ricci scalar of the world-sheet metric $\tilde{\gamma}$. Choosing a constant dilaton and using the Gauss–Bonnet theo-

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[1] There is interesting work on the quantum membrane and U-duality in string theory [6] which looks into quantum aspects of the membrane and the relation to string dualities.

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2006.02.038

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$$\frac{1}{4\pi}\int_{\Sigma} d^2\sigma\,\sqrt{\tilde{\gamma}}R^{(2)} = \chi(\Sigma) \tag{2}$$

reduces the action to

$$S_{\Phi} = \chi(\Sigma)\Phi. \tag{3}$$

Now, when one makes a Kałuża–Klein reduction of 11-dimensional supergravity on a circle of radius R_{11}, down to 10-dimensional type IIA supergravity, one finds that the dilaton is given by

$$e^{\Phi} = \left(\frac{R_{11}}{l_p}\right)^{3/2}. \tag{4}$$

Hence, in M-theory language,

$$g_s = \left(\frac{R_{11}}{l_p}\right)^{3/2}, \tag{5}$$

and after choosing units such that $l_p = 1$

$$S_{\Phi} = \frac{3}{2}\chi\log(R_{11}). \tag{6}$$

Both the absence of a factor of α' in the above action and the logarithmic dependence on R_{11} suggest that the origin of this term is not classical in nature. It is more likely that such a term arises from a quantum effective action. There is also the rather trivial observation that one would like to be able to *lift* the Euler character, $\chi(\Sigma)$ to three dimensions, so as to be able to give a membrane interpretation. Finding the correct quantity in three dimensions that when evaluated in $S^1 \times \Sigma$, where Σ is a Riemann surface, is therefore part of the puzzle.

The approach that we will adopt is to describe the fundamental string as a membrane with world volume topology $S^1 \times \Sigma$ with Σ being some Riemann surface. The membrane will be restricted to wrap once around the M-theory circle such that the world volume S^1 will be identified with the M-theory circle. We will truncate to the zero mode sector of the circle. That is there will be no dependence of any of the fields on circle direction, other than the winding mode. This is certainly justified if the M-theory circle is small since any excitations will be very heavy. One can also interpret this as isolating the pure fundamental string sector since any dependence on the M-theory circle will be associated with D0 branes in the string theory.

We will then describe the wrapped membrane partition function. In particular, we will be interested in calculating the measure for a membrane with the topology relevant for the string, that is $S^1 \times \Sigma$. We will then show that a careful treatment of this measure naturally gives rise to the correct dilaton coupling when interpreted from a string world sheet point of view.

2. The membrane

The bosonic part of the action for the M-theory 2-brane is, in Howe–Tucker form,

$$S_{\mathrm{M2}}[X,\gamma;G,C] = \frac{T_{\mathrm{M2}}}{2}\int_{M^3} d^3\sigma\,\sqrt{\gamma}\big(\gamma^{\mu\nu}\partial_\mu X^I\partial_\nu X^J G_{IJ} - 1 + \epsilon^{\mu\nu\rho}\partial_\mu X^I\partial_\nu X^J\partial_\rho X^K C_{IJK}\big), \tag{7}$$

where T_{M2} is the tension of the M2-brane, $\gamma_{\mu\nu}$ is the world-volume metric, $X^I(\sigma)$ specify the location of the brane in the target space. In what follows, upper case latin indices $I, J, K, \ldots$ are 11-dimensional target space indices whereas greek indices are world volume indices, and so σ^μ are the world volume coordinates. G_{IJ} and C_{IJK} are respectively the background metric and three form potential of eleven-dimensional supergravity. There are known obstacles for treating the membrane as one would the string. For example, there is no discrete spectrum of states that allows one to identify membrane states with space time excitations [7]. It is also not obviously a renormalisable theory in an arbitrary spacetime, even one that obeys the supergravity equations of motion. Yet there are various consistency checks, mostly from the supermembrane. κ symmetry of the supermembrane is consistent with the eleven-dimensional supergravity equations, [2]. After world volume dualisation of one the scalar fields, the membrane action can be identified with the D2 action [5]. The relation of the BPS sector of the membrane to dualities in string theory has been studied in [6] and important work on M-theory loops and higher derivative corrections to the string effective action corrections has been done in [8].

What we now propose, following analogous work of Polyakov for the string, is to take seriously the idea of a membrane partition function, Z given by:

$$Z = \sum_{\text{topologies}}\int\frac{\mathcal{D}X\,\mathcal{D}\gamma}{\mathrm{Vol}(\mathrm{Diff}_0)}e^{-S_{\mathrm{M2}}[X,\gamma;G,C]}, \tag{8}$$

or more precisely its supersymmetric extension. (For the present we will ignore the fermionic sector which appears to be irrelevant to our considerations regarding the origin of the dilaton coupling.) As with the string, the integrals are taken over all X^I and world volume metrics, $\gamma_{\mu\nu}$ and then summed over all topologies of the world volume. Since the action is diffeomorphism invariant one divides by the volume of three-dimensional diffeomorphisms. (For the string it would be the product of diffeomorphisms and Weyl transformations.)

To carry out the integral over world volume metrics, γ, one has to make an orthogonal decomposition of the deformations into those that are pure diffeomorphisms and those which are physical. The decomposition of the measure will then introduce a Jacobian which is the Faddeev–Popov determinant. One is then free to gauge fix and integrate over the pure diffeomorphism part of γ leaving only the physical degrees of freedom, the *moduli* of the three manifold. (The division by the volume of the diffeomorphism group then cancels the integral over the pure diffeomorphism part of γ.)

Thus after gauge fixing and integrating over pure diffeomorphisms one is left with the rather formal expression:

$$Z = \sum\int J\,d\{\mathrm{moduli}\}\,\mathcal{D}X\,e^{-S_{\mathrm{M2}}[X,\mathrm{moduli};G,C]}, \tag{9}$$