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On the gravitational energy of the Kaluza–Klein monopole

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Dedicated to Rafael Sorkin on the occasion of his 60th birthday

Abstract

We use local counterterm prescriptions for asymptotically flat space to compute the action and conserved quantities in five-dimensional Kaluza–Klein theories. As an application of these prescriptions we compute the mass of the Kaluza–Klein magnetic monopole. We find consistent results with previous approaches that employ a background subtraction.

1. Introduction

The problem of defining energy in theories involving gravity has a long-standing history. One would like for instance to be able to evaluate the total energy of an isolated object. Throughout the years many expressions have been proposed for computing the total energy. However, contrary to initial expectations, it was soon realised that finding satisfactory quantities is a very difficult task. The essential idea in computing the energy is to consider the values of the fields far away from the object and compare them with a background configuration, that is, with a 'no-fields situation'. This is for instance the approach considered when defining the ADM mass (see for instance [1]).

A related problem is that of computing the gravitational action of a non-compact spacetime. The gravitational action consists of the bulk Einstein–Hilbert term and it must be supplemented by the boundary Gibbons–Hawking term in order to have a well-defined variational principle. When evaluated on non-compact solutions of the field equations it turns out that both terms diverge. The general remedy for this situation is to consider the values of these quantities relative to those associated with some background reference spacetime, whose boundary at infinity has the same induced metric as that of the original spacetime. The background is chosen to have a topological structure that is compatible with that of the original spacetime and also one requires that the spacetimes approaches it sufficiently rapidly at infinity.

Unfortunately such background subtraction procedures are marred with difficulties: even if some choices of such reference background spaces present themselves as 'natural', in general these choices are by no means unique. Moreover, it is not always possible to embed a boundary with a given induced metric into the reference background and for different boundary geometries one needs different reference backgrounds [2]. A good example of the difficulties one might encounter in such an endeavour is that of the celebrated Taub-nut solution (see for instance [3–8]).

Similar difficulties and ambiguities are encountered when trying to compute the action and the conserved charges of the Kaluza–Klein monopole [9,10], and in particular its gravitational energy. Many such expressions for the conserved charges have been analysed in detail [12–14]; the consistent answers they yield when applied to the Kaluza–Klein monopole solution are for a definite choice of the reference background, one that is not a solution of the field equations. Moreover it is not a flat background, so that the energy expression for a Kaluza–Klein monopole is problematic. In general potential ambiguities arise in computing energy and other conserved quantities in

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dimensionally-reduced gravitation theories. This is partly because there are many distinct topological sectors, each of which requires a different background, and partly because within a given fixed topological sector, there may not be suitable background.

Motivated by recent results in the AdS/CFT conjecture, Balasubramanian and Kraus [15] proposed adding a term (referred to as a counterterm) to the boundary at infinity, which is a functional only of curvature invariants of the induced metric on the boundary. Such terms will not interfere with the equations of motion because they are intrinsic invariants of the boundary metric. By choosing appropriate counterterms, which cancel the divergences, one can then obtain well-defined expressions for the action and the energy momentum of the spacetime. Unlike background subtraction, this procedure is intrinsic to the spacetime of interest and is unambiguous once the counterterm is specified. While there is a general algorithm for generating the counterterms for asymptotically (A)dS spacetimes [16,17], the asymptotically flat case is considerably less-explored (see however [18] for some new results in this direction). Early proposals [19–22] engendered study of proposed counterterm expressions for a class of $(d+1)$-dimensional asymptotically flat solutions whose boundary topology is $S^n \times R^{d-n}$ [16]. This counterterm method has been applied to the five-dimensional black ring [23] and to an asymptotically Melvin spacetime [24].

The interesting properties of the Kaluza–Klein monopole merit further study in this context. In the present Letter we propose using a local counterterm prescription to compute its action and its conserved quantities in the five-dimensional Kaluza–Klein theory. In the next section we introduce the counterterm action and the expression for the conserved mass using the boundary stress-energy tensor. In the third section we apply this method to compute the action and the conserved mass of the Kaluza–Klein monopole from the five-dimensional point of view, while in the fourth section we compute the monopole energy from the four-dimensional perspective of the dimensionally reduced theory, using two distinct counterterm prescriptions. The last section is dedicated to conclusions, in which we comment on the relationships between the various approaches.

2. The counterterm action

In $(d+1)$-dimensions, the gravitational action is generally taken to be:

$$I_g = -\frac{1}{16\pi G}\int_M d^{d+1}x\,\sqrt{-g}R - \frac{1}{8\pi G}\int_{\partial M} d^d x\,\sqrt{-h}K. \tag{1}$$

Here M is a $(d+1)$-dimensional manifold with metric $g_{\mu\nu}$, K is the trace of the extrinsic curvature $K_{ij} = \frac{1}{2}h_i^k \nabla_k n_j$ of the boundary ∂M with unit normal n^i and induced metric h_{ij}.

For asymptotically flat four-dimensional spacetimes, the counterterm

$$I_{\rm ct} = \frac{1}{8\pi G}\int d^3x\,\sqrt{-h}\sqrt{2\mathcal{R}} \tag{2}$$

was proposed [19,20] to eliminate divergences that occur in (1). An analysis of the higher-dimensional case [16] suggested in 5 dimensions the counterterm

$$I_{\rm ct} = \frac{1}{8\pi G}\int d^4x\,\sqrt{-h}\frac{\mathcal{R}^{3/2}}{\sqrt{\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}}}, \tag{3}$$

where $\mathcal{R}_{ij}$ is the Ricci tensor of the induced metric h_{ij} and $\mathcal{R}$ is the corresponding Ricci scalar. This counterterm removes divergencies in the action for an asymptotically flat spacetime with boundary topology $S^3 \times R$ and also for an $S^2 \times R^2$ boundary topology.

By taking the variation of the action (3) with respect to the boundary metric h_{ij} we obtain

$$\begin{aligned}8\pi G(T_{\rm ct})^{ij} = {}& \frac{\mathcal{R}^{1/2}}{(\mathcal{R}^2 - \mathcal{R}_{kl}\mathcal{R}^{kl})^{3/2}}\big[3\mathcal{R}^{ij}\mathcal{R}_{kl}\mathcal{R}^{kl} - \mathcal{R}^{ij}\mathcal{R}^2 \\ & + 2\mathcal{R}\mathcal{R}^{ik}\mathcal{R}_k^j + \mathcal{R}^3 h^{ij} - \mathcal{R}\mathcal{R}_{kl}\mathcal{R}^{kl}h^{ij}\big] \\ & + \Phi^{(i}{}_k{}^{;j)k} - \frac{1}{2}\Box\Phi^{ij} - \frac{1}{2}h^{ij}\Phi^{kl}{}_{;kl},\end{aligned}$$

where:

$$\Phi^{ij} = \frac{\mathcal{R}^{1/2}}{(\mathcal{R}^2 - \mathcal{R}_{kl}\mathcal{R}^{kl})^{3/2}}\big[2\mathcal{R}\mathcal{R}^{ij} + \big(\mathcal{R}^2 - 3\mathcal{R}_{kl}\mathcal{R}^{kl}\big)h^{ij}\big],$$

so that the final boundary stress energy tensor is given by:

$$T_{ij} = \frac{1}{8\pi G}\big(K_{ij} - Kh_{ij} + (T_{\rm ct})_{ij}\big). \tag{4}$$

For a five-dimensional asymptotically flat solution with a fibred boundary topology $R^2 \hookrightarrow S^2$, we find that the action (1) can also be regularised using the following equivalent counterterm

$$I_{\rm ct} = \frac{1}{8\pi G}\int d^4x\,\sqrt{-h}\sqrt{2\mathcal{R}}, \tag{5}$$

where $\mathcal{R}$ is the Ricci scalar of the induced metric on the boundary, h_{ij}. By taking the variation of this total action with respect to the boundary metric h_{ij}, it is straightforward to compute the boundary stress-tensor, including (5):

$$T_{ij} = \frac{1}{8\pi G}\big(K_{ij} - Kh_{ij} - \Psi(\mathcal{R}_{ij} - \mathcal{R}h_{ij}) - h_{ij}\Box\Psi + \Psi_{;ij}\big),$$

where we denote $\Psi = \sqrt{\frac{2}{\mathcal{R}}}$. If the boundary geometry has an isometry generated by a Killing vector ξ^i, then $T_{ij}\xi^j$ is divergence free, from which it follows that the quantity

$$\mathcal{Q} = \oint_\Sigma d^3S^i\,T_{ij}\xi^j,$$

associated with a closed surface Σ, is conserved. Physically, this means that a collection of observers on the boundary with the induced metric h_{ij} measure the same value of $\mathcal{Q}$, provided the boundary has an isometry generated by ξ. In particular, if $\xi^i = \partial/\partial t$ then $\mathcal{Q}$ is the conserved mass $\mathcal{M}$.

The counterterm (3) was proposed in [16] for spacetimes with boundary $S^2 \times R^2$, or $S^3 \times R$. On the other hand, the counterterm (3) is essentially equivalent to (5) for $S^2 \times R^2$ boundaries. We find that when the boundary is taken to infinity both expressions cancel the divergences in the action. Our