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Quantum Liouville theory on the pseudosphere with heavy charges ☆

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Abstract

We develop a perturbative expansion of quantum Liouville theory on the pseudosphere around the background generated by heavy charges. Explicit results are presented for the one and two point functions corresponding to the summation of infinite classes of standard perturbative graphs. The results are compared to the one point function and to a special case of the two point function derived by Zamolodchikov and Zamolodchikov in the bootstrap approach, finding complete agreement. A partial summation of the conformal block is also obtained.

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Much interest has been devoted to the exact solutions of the conformal bootstrap equations for the Liouville theory on the pseudosphere [1], on the finite disk with conformally invariant boundary conditions [2] and on the sphere [3–5]. The first two solutions have given rise to the ZZ and FZZT branes.

In this Letter, we present a technique to treat the Liouville field theory on the pseudosphere, which allows to find an expansion in the coupling constant b of the N point functions in presence of "heavy charges", according to the terminology introduced in [3]. This means that we consider the vertex operator $V_\alpha(z) = e^{2\alpha\phi(z)}$ with $\alpha = \eta/b$ and η fixed in the semiclassical limit $b \to 0$.

For the one point function, this analysis goes well beyond the previous perturbative expansion performed in [1,6,7] where α has been taken small; indeed, our result corresponds to the summation of an infinite class of perturbative graphs. Thus, we obtain a strong check of the ZZ bootstrap formula for the one point function [1], which includes all the previous perturbative checks.

We apply the same technique to compute the two point function with two heavy charges η and ε, to the first order in ε, getting a closed expression of this correlator to the orders $O(b^{-2})$ and $O(b^0)$ included, but exact in η and in the $SU(1,1)$ invariant distance ω between the sources. According to an argument given by ZZ, such expression provides an expansion of a conformal block up to $O(b^0)$ and to the first order in ε, but to all orders in η and ω. With more work, this technique can be extended to higher orders in b^2 and to more complicated correlation functions.

We start from the Liouville action on the pseudosphere in presence of N sources characterized by heavy charges $\eta_1, \dots, \eta_N$, given in [7]. Decomposing the Liouville field ϕ as follows

$$\phi = \frac{1}{2b}(\varphi_B + 2b\chi) \tag{1}$$

the Liouville action separates into a classical part, depending only on the background field φ_B, and a quantum action for the

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quantum field χ

$$S_{\Delta,N}[\phi] = S_{\rm cl}[\varphi_B] + S_q[\varphi_B, \chi]. \tag{2}$$

Adopting the unit disk representation $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$ for the pseudosphere, these actions read

$$\begin{aligned} S_{\rm cl}[\varphi_B] &= \frac{1}{b^2} \lim_{\substack{\varepsilon\to 0\\ r\to 1}} \left\{ \int_{\Delta_{r,\varepsilon}} \left[\frac{1}{4\pi}\partial_z\varphi_B\partial_{\bar z}\varphi_B + \mu b^2 e^{\varphi_B} \right] d^2 z \right. \\ &\quad - \frac{1}{4\pi i}\oint_{\partial\Delta_r} \varphi_B\left(\frac{\bar z}{1-z\bar z}dz - \frac{z}{1-z\bar z}d\bar z\right) + f_{\rm cl}\big(r,\mu b^2\big) \\ &\quad \left. - \frac{1}{4\pi i}\sum_{n=1}^{N}\eta_n \oint_{\partial\gamma_n}\varphi_B\left(\frac{dz}{z-z_n} - \frac{d\bar z}{\bar z - \bar z_n}\right) - \sum_{n=1}^{N}\eta_n^2\log\varepsilon_n^2 \right\} \end{aligned} \tag{3}$$

and

$$\begin{aligned} S_q[\varphi_B,\chi] &= \lim_{r\to 1}\left\{ \int_{\Delta_r}\left[\frac{1}{\pi}\partial_z\chi\,\partial_{\bar z}\chi + \mu e^{\varphi_B}\big(e^{2b\chi} - 1 - 2b\chi\big)\right] d^2 z \right. \\ &\quad \left. - \frac{b}{2\pi i}\oint_{\partial\Delta_r}\chi\left(\frac{\bar z}{1-z\bar z}dz - \frac{z}{1-z\bar z}d\bar z\right)\right\}, \end{aligned} \tag{4}$$

where the function $f_{\rm cl}(r,\mu b^2)$ is a subtraction term independent of the charges. The coupling constant b is related to the parameter Q occurring in the central charge $c = 1 + 6Q^2$ by $Q = 1/b + b$ [8]. Moreover, the classical field φ_B obeys the following boundary conditions

$$\varphi_B(z) = -\log(1-z\bar z)^2 + f\big(\mu b^2\big) + O\big((1-z\bar z)^2\big), \quad \text{when } |z| \to 1, \tag{5}$$

$$\varphi_B(z) = -2\eta_n \log|z - z_n|^2 + O(1), \quad \text{when } z \to z_n, \tag{6}$$

where $f(\mu b^2)$ is a function depending only on the product μb^2. The integration domains are $\Delta_{r,\varepsilon} = \Delta_r \setminus \bigcup_{n=1}^{N}\gamma_n$ with $\Delta_r = \{|z| < r < 1\} \subset \Delta$ and $\gamma_n = \{|z - z_n| < \varepsilon_n\}$. Because of the boundary behavior of the Green function, that will be computed in the following, the last integral in the quantum action (4) does not contribute.

The vanishing of the first variation of $S_{\rm cl}[\varphi_B]$ with respect to the field φ_B satisfying (5) and (6) gives the Liouville equation in presence of N sources

$$-\partial_z\partial_{\bar z}\varphi_B + 2\pi b^2\mu e^{\varphi_B} = 2\pi\sum_{n=1}^{N}\eta_n\delta^2(z - z_n). \tag{7}$$

At semiclassical level, we have

$$\big\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)\big\rangle_{\rm sc} = \frac{e^{-S_{\rm cl}(\eta_1,z_1;\dots;\eta_N,z_N)}}{e^{-S_{\rm cl}(0)}}, \tag{8}$$

where $S_{\rm cl}(\eta_1, z_1; \dots; \eta_N, z_N)$ is the classical action $S_{\rm cl}[\varphi_B]$ computed on the solution φ_B of the Liouville equation with sources. Now, since under a $SU(1,1)$ transformation the classical background field changes as follows

$$\varphi_B(z) \longrightarrow \tilde\varphi_B(w) = \varphi_B(z) - \log\left|\frac{dw}{dz}\right|^2 \tag{9}$$

one can see that the transformation law of $S_{\rm cl}[\varphi_B]$ assigns to the vertex operator $V_\alpha(z)$ the semiclassical dimensions $\alpha(1/b - \alpha) = \eta(1-\eta)/b^2$ [7,9].

For the one point function, we have a single heavy charge $\eta_1 = \eta$, which can be placed in $z_1 = 0$, and the explicit solution of the Liouville equation is $\varphi_B = \varphi_{\rm cl}$, given by [10]

$$e^{\varphi_{\rm cl}} = \frac{1}{\pi\mu b^2}\frac{(1-2\eta)^2}{[(z\bar z)^\eta - (z\bar z)^{1-\eta}]^2}. \tag{10}$$

Local finiteness of the area around the source in the metric $e^{\varphi_{\rm cl}}\,d^2 z$ imposes $\eta < 1/2$ [10,11].

The classical action (3) computed on this background gives the semiclassical one point function

$$\big\langle V_{\eta/b}(0)\big\rangle_{\rm sc} = \exp\left\{-\frac{1}{b^2}\big(\eta\log\big[\pi b^2\mu\big] + 2\eta + (1-2\eta)\log(1-2\eta)\big)\right\}. \tag{11}$$

To go beyond this approximation, we need to find the Green function on the background field given by (10) and, to do this, we employ the method developed in [12]. Thus, we compute the classical background in presence of the charge η in $z_1 = 0$ and of another charge ε in $z_2 = t \in \Delta$ and real by applying first order perturbation theory in ε to the Fuchsian equation associated to the Liouville equation, i.e.

$$\frac{d^2 Y_j}{dz^2} + Q(z)Y_j = 0, \qquad j = 1, 2, \tag{12}$$

where $Q(z) = b^2 T(z)$, being $T(z)$ the holomorphic component of the classical energy momentum tensor of the classical field φ_B. To first order perturbation theory in ε, one writes $Y_j = y_j + \varepsilon\delta y_j$ and $Q = Q_0 + \varepsilon q$, where the unperturbed quantities are $Q_0(z) = \eta(1-\eta)/z^2$, $y_1(z) = z^\eta$ and $y_2(z) = z^{1-\eta}$.

We impose now the Cardy condition [13] and the regularity condition at infinity on the classical energy momentum tensor. One can express them more easily in the upper half plane $\mathbb{H} = \{\xi \in \mathbb{C};\ \mathrm{Im}(\xi) > 0\}$ representation (related to the unit disk representation through the Cayley transformation $z = (\xi - i)/(\xi + i)$), where they read $\tilde Q(\xi) = \bar{\tilde Q}(\xi)$ and $\xi^4\tilde Q(\xi) \sim O(1)$ when $\xi \to \infty$, respectively.

In the $\mathbb{H}$ representation, we have that $\tilde Q_0(\xi) = 4\eta(\eta - 1)/(\xi^2+1)^2$ and

$$\tilde q(\xi) = \frac{1}{(\xi - i\tau)^2} + \frac{1}{(\xi + i\tau)^2} + \frac{\beta_i}{2(\xi - i)} + \frac{\bar\beta_i}{2(\xi + i)} + \frac{\beta_{i\tau}}{2(\xi - i\tau)} + \frac{\bar\beta_{i\tau}}{2(\xi + i\tau)}, \tag{13}$$

where β_i and $\beta_{i\tau}$ are the Poincaré accessory parameters and $i\tau = i(1+t)/(1-t)$ is the image of $z_2 = t$ through the Cayley transformation.