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Q-operator and T–Q relation from the fusion hierarchy

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Abstract

We propose that the Baxter Q-operator for the spin-1/2 XXZ quantum spin chain is given by the $j \to \infty$ limit of the transfer matrix with spin-j (i.e., $(2j+1)$-dimensional) auxiliary space. Applying this observation to the open chain with general (non-diagonal) integrable boundary terms, we obtain from the fusion hierarchy the T–Q relation for *generic* values (i.e., not roots of unity) of the bulk anisotropy parameter. We use this relation to determine the Bethe ansatz solution of the eigenvalues of the fundamental transfer matrix. This approach is complementary to the one used recently to solve the same model for the roots of unity case.

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1. Introduction

The Baxter Q-operator is a fundamental object in the theory of exactly solvable models [1]. Nevertheless, it has been an enigma. Indeed, while the transfer matrix has a systematic construction in terms of solutions of the Yang–Baxter equation, the Q-operator's original construction—its brilliance notwithstanding—was ad hoc. In particular, the Q-operator seemed to be absent from the quantum inverse scattering method (QISM). It was later understood [2,3] that the Q-operator could be realized by a transfer matrix whose associated auxiliary space is infinite dimensional. However, its relation to the QISM remained unclear.

Motivated in part by [2,3], we propose here that the Q-operator $\bar{\mathcal{Q}}(u)$ for a spin-1/2 XXZ quantum spin chain is given by the $j \to \infty$ limit of the transfer matrix $t^{(j)}(u)$ with spin-j (i.e., $(2j+1)$-dimensional) auxiliary space,

$$\bar{\mathcal{Q}}(u) = \lim_{j\to\infty} t^{(j)}(u - 2j\eta), \tag{1.1}$$

where η is the anisotropy parameter. This relation makes it clear that the Q-operator does in fact fit naturally within the QISM. Moreover, this relation together with the fusion hierarchy for the closed-chain transfer matrix [4–7]

$$\begin{aligned} &t^{(\frac{1}{2})}(u)\, t^{(j)}(u-2j\eta) \\ &\quad = \sinh^N(u+\eta)\sinh^N(u-\eta) t^{(j-\frac{1}{2})}\left(u - 2\left(j-\frac{1}{2}\right)\eta - \eta\right) + t^{(j+\frac{1}{2})}\left(u - 2\left(j+\frac{1}{2}\right)\eta + \eta\right), \quad j = \frac{1}{2}, 1, \frac{3}{2}, \ldots \end{aligned} \tag{1.2}$$

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immediately leads to the Baxter T–Q relation

$$t^{(\frac{1}{2})}(u)\bar{\mathcal{Q}}(u) = \sinh^N(u+\eta)\sinh^N(u-\eta)\bar{\mathcal{Q}}(u-\eta) + \bar{\mathcal{Q}}(u+\eta), \tag{1.3}$$

from which it is possible to derive the well-known expression for the eigenvalues of the fundamental transfer matrix $t^{(\frac{1}{2})}(u)$ and the associated Bethe ansatz equations. However, we emphasize that the above argument is formal: we assume without proof that the limit in (1.1) exists, and we do not evaluate the right-hand side explicitly.

It is interesting to apply this observation to the open spin-1/2 XXZ quantum spin chain with general integrable boundary terms [8,9]. Indeed, this model remains unsolved, although the special case of diagonal boundary terms was solved long ago [10–12]. Significant progress has been made recently for the case of non-diagonal boundary terms where the boundary parameters obey some constraints. One approach [13] (see also [14]) is based on the generalized algebraic Bethe ansatz [15,16].[1] A second approach, which was developed in [17], exploits functional relations obeyed by the transfer matrix at roots of unity to obtain the eigenvalues of the transfer matrix. These functional relations are a consequence of the truncation of the fusion hierarchy of the transfer matrix at roots of unity [19,20].

In this Letter, we develop a third approach, which is complementary to the second one [17]. Indeed, as in [17], we make use of the fusion hierarchy for the open XXZ chain [21,22]. However, here we consider instead *generic* values (i.e., not roots of unity) of the bulk anisotropy parameter, for which the fusion hierarchy does *not* truncate. We instead use the relation (1.1) to obtain the T–Q relation. We then use the latter relation, together with some additional properties of the transfer matrix, to determine the eigenvalues of the transfer matrix and the associated Bethe ansatz equations. The expressions for the eigenvalues are generalizations of those found in [17], and this new derivation explains their validity for generic anisotropy values [18].[2]

The Letter is organized as follows. In Section 2, we introduce our notation and some basic ingredients. In Section 3, we derive the T–Q relation from (1.1) and the fusion hierarchy of the open XXZ chain. Through that relation, we determine the eigenvalues of the transfer matrix and the associated Bethe ansatz equations in Section 4. Finally, we summarize our conclusions and mention some interesting open problems in Section 5.

2. Transfer matrix

Throughout, let us fix a generic complex number η, and let $\sigma^x, \sigma^y, \sigma^z$ be the usual Pauli matrices. The well-known six-vertex model R-matrix $R(u) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is given by

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh(u) & \sinh(\eta) & \\ & \sinh(\eta) & \sinh(u) & \\ & & & \sinh(u+\eta) \end{pmatrix}. \tag{2.1}$$

Here u is the spectral parameter and η is the so-called bulk anisotropy parameter. The R-matrix satisfies the quantum Yang–Baxter equation and the properties,

$$\text{Unitarity relation:}\quad R_{1,2}(u)R_{2,1}(-u) = -\xi(u)\mathrm{id}, \quad \xi(u) = \sinh(u+\eta)\sinh(u-\eta), \tag{2.2}$$

$$\text{Crossing relation:}\quad R_{1,2}(u) = V_1 R_{1,2}^{t_2}(-u-\eta)V_1, \quad V = -i\sigma^y, \tag{2.3}$$

$$\text{Periodicity property:}\quad R_{1,2}(u+i\pi) = -\sigma_1^z R_{1,2}(u)\sigma_1^z. \tag{2.4}$$

Here $R_{2,1}(u) = P_{12}R_{1,2}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes transposition in the ith space. Here and below we adopt the standard notations: for any matrix $A \in \mathrm{End}(\mathbb{C}^2)$, A_j is an embedding operator in the tensor space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$, which acts as A on the jth space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the ith and jth ones.

The transfer matrix $t(u)$ of the open XXZ chain with general integrable boundary terms is given by [12]

$$t(u) = \mathrm{tr}_0\big(K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\big), \tag{2.5}$$

where $T_0(u)$ and $\hat{T}_0(u)$ are the monodromy matrices

$$T_0(u) = R_{0,N}(u)\cdots R_{0,1}(u), \qquad \hat{T}_0(u) = R_{1,0}(u)\cdots R_{N,0}(u), \tag{2.6}$$

and tr_0 denotes trace over the "auxiliary space" 0. We consider the most general solutions $K^{\mp}(u)$ [8,9] to the reflection equation and its dual [12,24]. The matrix elements are given respectively by

[1] It is not yet clear whether this approach can give all the eigenvalues of the transfer matrix. Indeed, in order to obtain all the levels, two sets of Bethe ansatz equations are required [18], and therefore, presumably two pseudovacua. However, it is not yet clear how to construct the second pseudovacuum.

[2] Additional solutions have recently been found at roots of unity which are *not* valid for generic anisotropy values [23].