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The Casimir effect in the presence of a minimal length

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Abstract

Large extra dimensions could lower the Planck scale to experimentally accessible values. Not only is the Planck scale the energy scale at which effects of modified gravity become important. The Planck length also acts as a minimal length in nature, providing a natural ultraviolet cutoff and a limit to the possible resolution of spacetime.

In this Letter we examine the influence of the minimal length on the Casimir energy between two plates.

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1. Extra dimensions

The study of models with large extra dimensions (LXDs) has recently received a great deal of attention. These models, which are motivated by string theory [1–3], provide us with an extension to the standard model (SM) in which observables can be computed and predictions for tests beyond the SM can be addressed. This in turn might help us to extract knowledge about the underlying theory. The models of LXDs successfully fill the gap between theoretical conclusions and experimental possibilities as the extra hidden dimensions may have radii large enough to make them accessible to experiments. The need to look beyond the SM infected many experimental groups to search for such SM violating processes, for a summary see, e.g., [4]. In this Letter we will work within an extension of the LXD-model [5–8] (for recent constraints see [9]) that self-consistently includes a minimal length scale. Since the LXDs result in a lowered fundamental scale, also the minimal length might get observable soon and we should clearly take into account the arising effect.

2. The minimal length

In perturbative string theory [10,11], the feature of a fundamental minimal length scale arises from the fact that strings cannot probe distances smaller than the string scale. If the energy of a string reaches this scale $M_s = \sqrt{\alpha'}$, excitations of the string can occur and increase its extension [12]. In particular, an examination of the spacetime picture of high-energy string scattering shows that the extension of the string grows proportional to its energy [10] in every order of perturbation theory. Due to this, uncertainty in position measurement can never become arbitrarily small.

Motivations for the occurrence of a minimal length are manifold. A minimal length cannot only be found in string theory [10–12] but also in loop quantum gravity and non-commutative geometries. It can be derived from various studies of thought-experiments, from investigations of the Heisenberg–Poincaré algebra [13], from black hole physics, the holographic principle and further more. Perhaps the most convincing argument, however, is that there seems to be no self-consistent way to avoid

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the occurrence of a minimal length scale. The minimal length acts as a regulator in the ultra violet and seems to be necessary for our understanding of physics near the Planck scale. For reviews on this topic see, e.g., [14].

Instead of finding evidence for the minimal scale as has been done in numerous studies, on can use its existence as a postulate and derive extensions to quantum theories [15] with the purpose to examine the arising properties in an effective model.

In [16,17] a model for the minimal length has been worked out, which includes the new effects by modifying the relation between the wave vector k and the momentum p. It is assumed that, no matter how much the momentum p of a particle is increased, its wavelength can never be decreased below some minimal length L_f or, equivalently, its wave-vector k can never be increased above $M_\mathrm{f} = 1/L_\mathrm{f}$ [18]. Thus, the relation between the momentum p and the wave vector k is no longer linear $p = k$ but a function[1] $k = k(p)$.

This function $k(p)$ has to fulfill the following properties:

(a) For energies much smaller than the new scale we reproduce the linear relation: for $p \ll M_\mathrm{f}$ we have $p \approx k$.
(b) It is an uneven function (because of parity) and $k \parallel p$.
(c) The function asymptotically approaches the upper bound M_f.

The quantization in this scenario is straightforward and follows the usual procedure. The commutators between the corresponding operators $\hat{k}$ and $\hat{x}$ remain in the standard form whereas the functional relation between the wave vector and the momentum then yields the modified commutator for the momentum

$$[\hat{x}_i, \hat{p}_j] = +\mathrm{i}\frac{\partial \hat{p}_i}{\partial \hat{k}_j}, \tag{1}$$

where the derivative is the quantized version of $\partial p_i/\partial k_j$, most easily to be interpreted in the polynomial series expansion.[2] This then results in the generalized uncertainty relation (GUP)

$$\Delta p_i \Delta x_j \geqslant \frac{1}{2}\left|\left\langle \frac{\partial p_i}{\partial k_j}\right\rangle\right|, \tag{2}$$

which reflects the fact that by construction it is not possible anymore to resolve space–time distances arbitrarily good. Since $k(p)$ gets asymptotically constant, its derivative $\partial k/\partial p$ drops to zero and the uncertainty in Eq. (2) increases for high energies. Thus, the introduction of the minimal length reproduces the limiting high energy behavior found in string theory [10].

The arising physical modifications—as investigated in [16, 17,19]—can be traced back to an effective replacement of the usual momentum measure by a measure which is suppressed at high momenta:

$$\frac{\mathrm{d}^3 p}{(2\pi)^3} \to \frac{\mathrm{d}^3 p}{(2\pi)^3}\left|\frac{\partial k}{\partial p}\right|, \tag{3}$$

where the absolute value of the partial derivative denotes the Jacobian determinant of $k(p)$. Here, the left side of the replacement Eq. (3) is the standard expression, whereas the right side is the modified version as arises from the inclusion of the minimal length scale. In k-space, the modification translates into a finiteness of the integration bounds.

The exact form of the functional relation $k(p)$ is unknown but it is strongly constrained by the above listed requirements (a)–(c); in the literature various choices have been used. The exact form of the functional relation will make a quantitative difference in the range where the first deviations from the linear behavior become important. These can, e.g., be parametrized in a polynomial expansion. However, in the large p-limit, the requirement (c) will lead to a convergence of all functions. Though the intermediate region would be important for the quantitative examination, we will here be interested in making a qualitative statement, dominated by the assumed asymptotic behavior.

In the following, we will use the specific relation from [17] for $k(p)$ by choosing

$$k_\mu(p) = \hat{e}_\mu \int_0^p e^{-\epsilon p'^2}\,\mathrm{d}p', \tag{4}$$

where $\hat{e}_\mu$ is the unit vector in μ-direction, $p^2 = \vec{p}\cdot\vec{p}$ and $\epsilon = L_\mathrm{f}^2\pi/4$ (the factor $\pi/4$ is included to assure, that the limiting value is indeed $1/L_\mathrm{f}$). It is easily verified that this expression fulfills the requirements (a)–(c).

The Jacobian determinant of the function $k(p)$ is best computed by adopting spherical coordinates and can be approximated for $p \sim M_\mathrm{f}$ with

$$\left|\frac{\partial k}{\partial p}\right| \approx e^{-\epsilon p^2}. \tag{5}$$

With this parametrization of the minimal length effects, the modifications read

$$\Delta p_i \Delta x_i \geqslant \frac{1}{2} e^{+\epsilon p^2}, \tag{6}$$

$$\frac{\mathrm{d}^3 p}{(2\pi)^3} \to \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-\epsilon p^2}. \tag{7}$$

In field theory,[3] one imposes the commutation relation Eq. (1) on the field ϕ and its conjugate momentum Π. Its Fourier expansion leads to the annihilation and creation operators which must obey

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = -\mathrm{i}[\hat{\phi}_k, \hat{\Pi}_{k'}^\dagger], \tag{8}$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta(k - k'), \tag{9}$$

$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = e^{-\epsilon p^2}\delta(p - p') \tag{10}$$

(see also Ref. [16]).

Note, that it is not necessary for our field to propagate into the extra dimensions to experience the consequences of the minimal length scale. In particular, we will assume that the field

[1] Note, that this is similar to introducing an energy dependence of Planck's constant $\hbar$.

[2] There is no arbitrariness in the quantization since p is not a function of x by assumption.

[3] For simplicity, we consider a massless scalar field.